Proofs of the inscribed angle theorem

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Abstract

The inscribed angle theorem says that central angle is double of an inscribed angle when the angles have the same arc of base. It is traditionally proved by the same way as Euclid in his Elements introduced, although a simpler and more modern ways are possible. The contribution shows two of them by Cabri tools representation. Files can be used in interpreting the curriculum or for self-discovery work of pupils.

The first proof uses properties of axis of chords and its corollary is a proposition which says that composition of reflections over two intersecting lines is rotation.

In the second proof is firstly derived theorem of intersecting chords which says that double angle formed inside by two non parallel chords in a circle is equal to sum of intercepted arcs. The proposition is easy to show by symmetry of arcs between parallel chords. The inscribed angle theorem is a corollary of the intersecting chords proposition.

Keywords

Secondary school geometry, circle, inscribed angle theorem, Cabri geometry.

1 Introduction

The current elementary geometry is still taught largely in accordance with Euclid Elements. One of the basic propositions is the inscribed angle theorem placed in the Elements as prop. III.20. It says that central angle is double of an inscribed angle when the angles have the same arc of base. Also proof of a theorem we do like Euclid, with the only difference: Euclid took one diagram (see Fig. 1) and now we usually distinguish five different situations (Fig. 2).

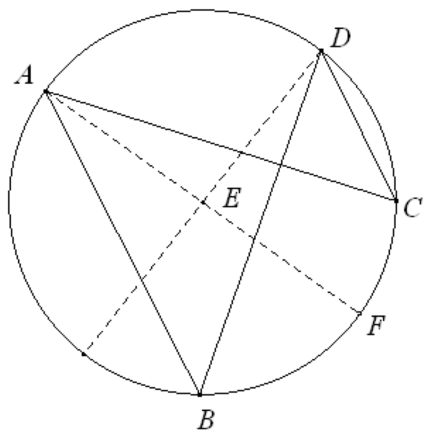

Figure 1

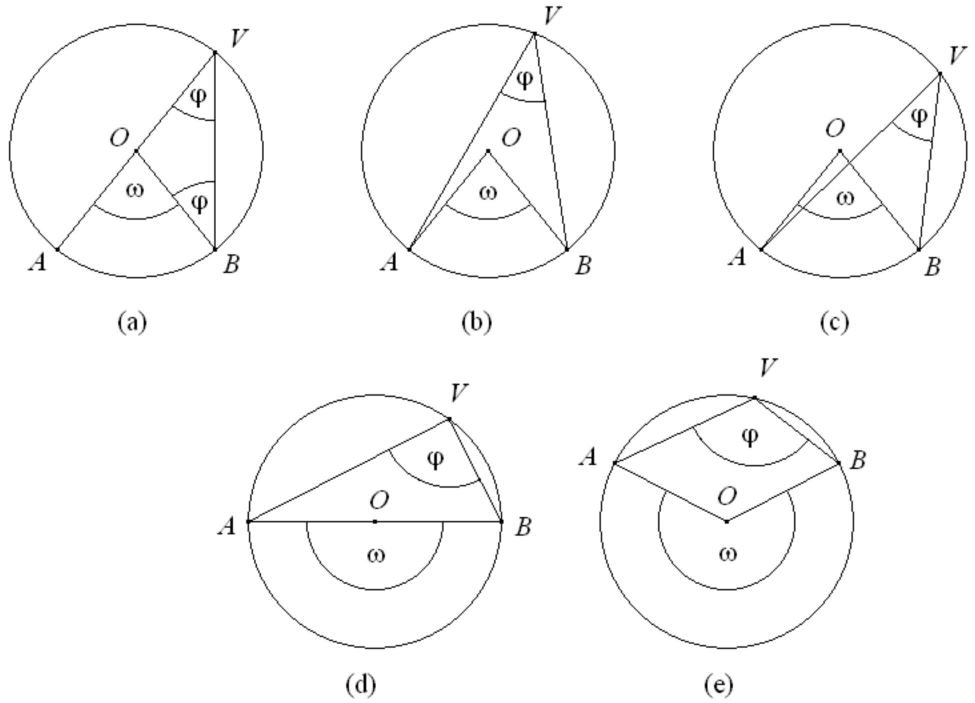

Figure 2

In the first situation, the proposition follows from the properties of exterior angle AOB of the isosceles triangle OBV: $\omega = 2\varphi,$ see Fig. 2 (a). The other situations may be transformed to prior by adding line OV. An interactive file for examining all the situation, forming conclusions and evidence is shown in Figure 3.

As we can see, a complete proof is somewhat lengthy. The question is whether there are more effective methods of teaching the curriculum. Two alternative ways are presented below as activities with the support of dynamic geometry.

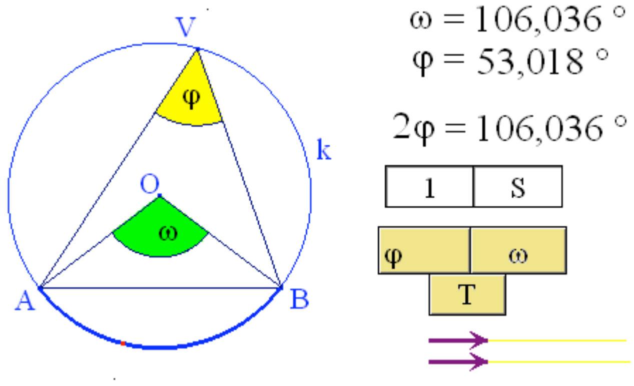

Figure 3

2 Axis of chords method

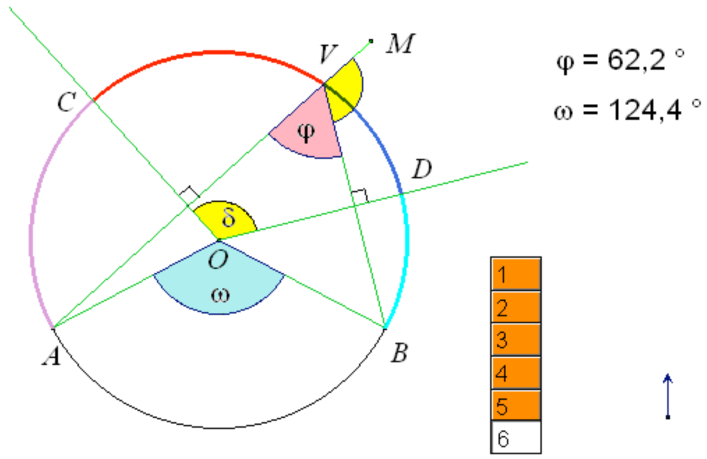

Figure 4

A simple proof of the inscribed angle theorem can be built on the both: fact that axis of each chord contains the center of the circle and fact that perpendiculars to the two lines are formed by the same angles as those lines. In Figure 4 axis OC divides the arc AV into two congruent arcs AC and CV. Analogously the axis OD divides the arc BV into congruent arcs BD and DV. Hence the arc CD represents half of the arc AB and $|\angle BVM| = |\angle COD| = \delta.$ From the last expression and Figure 4 follows

$$\omega = 2\pi - 2\delta = 2(\pi - \delta) = 2\varphi. \quad (1)$$

Note that similar way mentioned Lietzmann (1951). Formula (1) also follows from the fact that the composition of reflections in the lines OC and OD is a rotation by 2φ about center O (Kuřina 2004).

Figure 4 also presents a dynamic tool created in Cabri for this way of teaching or for student's creative work. Using the vector slider can be change the position of the chord AB and thus sizes of ω and $\varphi.$ Point V can be moved on a circle with the mouse. Buttons allow you to hide or show parts of the figure. Button 1 allows hiding or showing of axis

OC and OD. Button 2 is for hiding or showing of marks of right angles, buttons 3 and 4 are for hiding or showing of arcs CV, DV and AC, BD. Button 5 is for marks of angles COD and BVM. Button 6 is for hiding or showing of expression (1).

3 Theorem of intersecting chords

Fig. 5 shows an aid created in Cabri for exploring geometric situations with intersecting chords. Pupils carry out experiments to create a hypothesis how the sum of the lengths of arcs AB and CD depends on the position chords AC and BD.

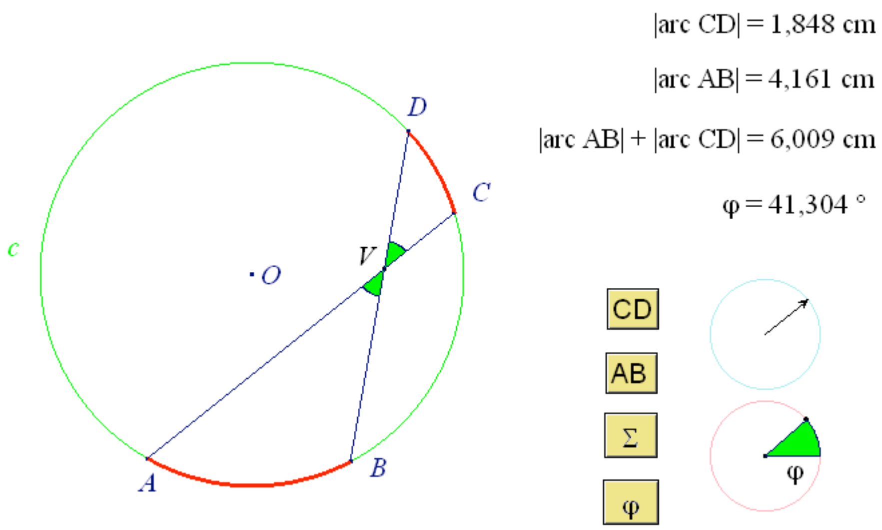

Figure 5

Four buttons allow you to hide and show data on the actual arch lengths, the sum of these lengths and the size of the angle φ marked in green. In grasping the point V can be changed its position inside the circle. The upper circle slider is able to rotate chord AC and BD at about the point V without changing the size of their angle. By the lower slider we can change the measure of angle φ. By grabbing at any unmarked circle point we can change its size.

Work with this file leads to the hypothesis that in given circle the sum of the lengths of arcs AB and CD remains constant by changing chords while keeping the measure of angle φ. Student may find evidence of the hypothesis himself by another file (see Figure 6a and 6b).

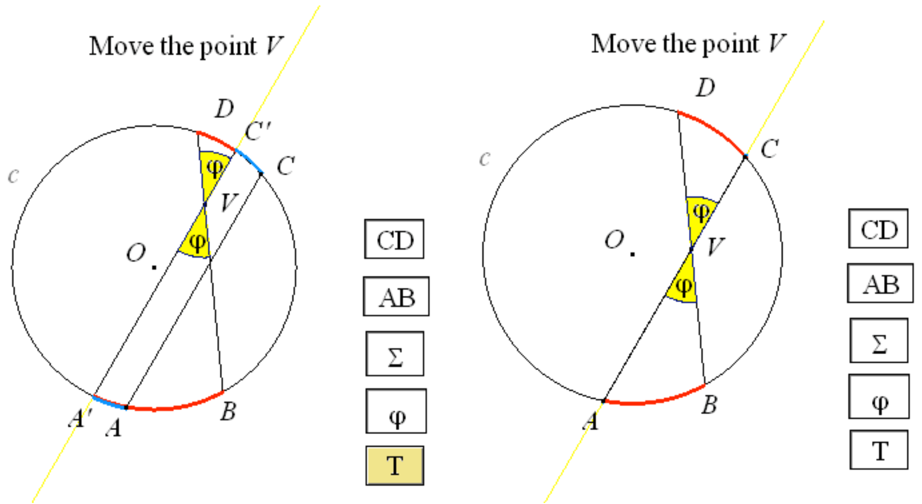

Figure 6a / 6b

In Figure 6a we can see the initial state after opening the file. By grabbing at the point V we can move the chord AC to a position $A'C'$. After the transfer, the arc CD is reduced by the CC' and the arc AB is extended by the AA'. But arcs CC' and AA' are congruent because of reflection over common axis of parallel chords AC and $A'C'$. So the sum of the lengths of arcs AB and CD remains constant. If the pupils do not appear it themselves, can be used as a help file in Figure 7.

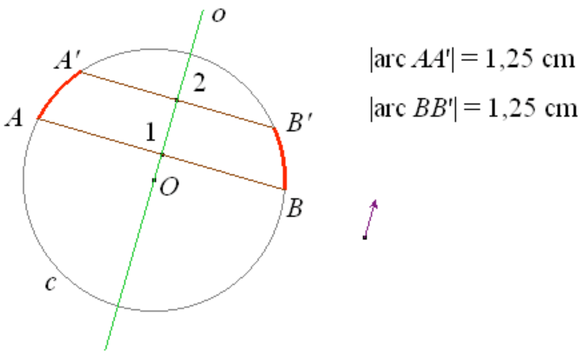

Figure 7.

Determination of the value of this sum allows the activity with a next file (see Figure 8a, b). In Figure 8a is the initial state after opening the file. In Figure 8b is the situation after the moving.

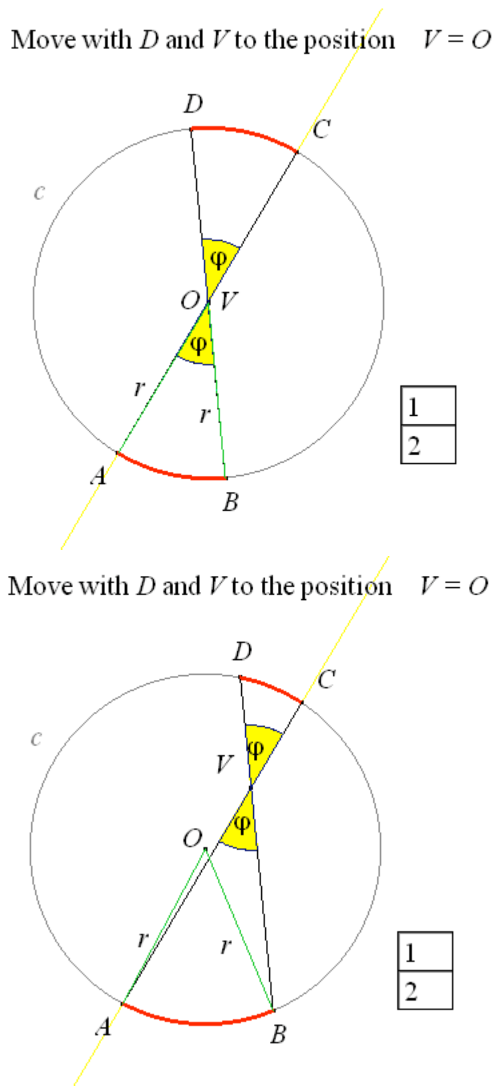

Figure 8a / 8b

Figure 8b shows that $|\text{arc } AB| = r\varphi = |\text{arc } CD|$, whence

$$|\text{arc } AB| + |\text{arc } CD| = 2r\varphi. \quad (2)$$

The last relations could be shown or hidden by buttons 1 and 2 in the file (see Figure 8a, b).

Previous considerations can be summarized in the following proposition.

Theorem of intersecting chords. For each circle the product of angle formed inside by two chords and double of the radius of circle is equal to sum of lengths of intercepted arcs.

Note that the theorem can be generalized to the chords whose secants intersect outside the circle (Ponarin 2004)

Consistent with Figure 9, we can rewrite formula (2) in the shape $r\omega + r\omega' = 2r\varphi$ of which, divided by r, we get

$$\omega + \omega' = 2\varphi \quad (3)$$

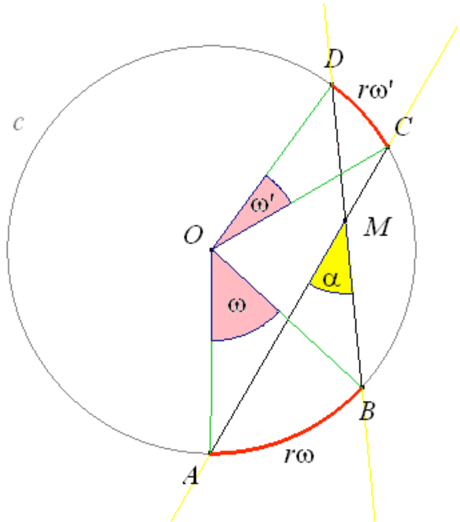

Figure 9

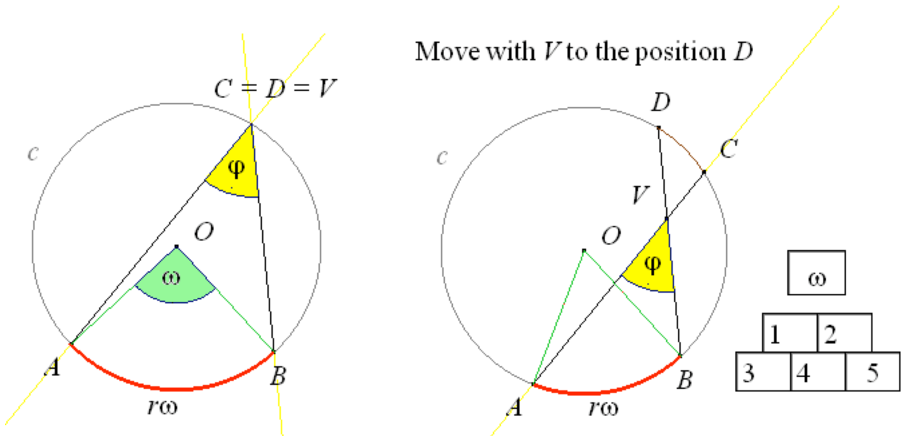

Figure 10a / 10b

The inscribed angle theorem is a corollary of intersecting chords proposition. Pupils can find it using next file. The Figure 10a shows the file after opening. Figure 10b) shows the situation after moving. The mark of central angle can be shown or hidden by button ω. The other buttons can show or hide a mathematical derivation of the inscribed angle formula which follows from the Figure 10b and equation (2):

$$|\text{arc } AB| = r\omega, |\text{arc } CD| = 0 \Rightarrow r\omega = |\text{arc } AB| + |\text{arc } CD| = 2r\varphi \Rightarrow \omega = 2\varphi.$$

4 Some applications

Solving of some geometric problems is easier by theorem of intersecting chords than by the inscribed angle theorem. We show that in several examples.

On the clock face dial are constructed three lines that define the triangle ABC in Figure 11. Determine angles α, β and γ of the triangle.

Solution. The dial divides its circle into twelve congruent arcs with central angle 30^0. From the relation (3) and Figure 12 we get

$2\alpha = \omega_1 + \omega_4 = 30^0 + 90^0,\ 2\beta = \omega_2 + \omega_5 = 60^0 + 30^0$ and

$2\gamma = \omega_3 + \omega_6 = 30^0 + 120^0$. Whence $\alpha = 60^0,\ \beta = 45^0$ and $\gamma = 75^0$.

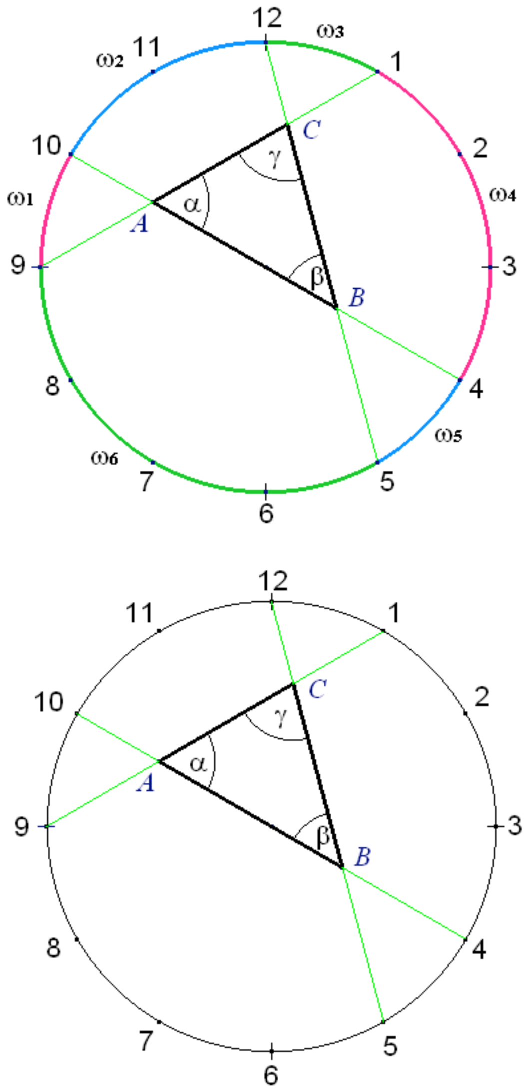

Figure 11 / 12

Quadrilateral ABCD is inscribed in the circle. Denote K, L, M, N center of the no overlapping arcs AB, BC, CD, DA respectively (Figure 13). Prove that chords KM and LN are perpendicular.

Solution. Let radius of circle $r = 1,$ then the arc length is equal to the corresponding central angle (in radian measure). From the relation (3) and Figure 12 we get

$(\omega_1 + \omega_4) + (\omega_2 + \omega_3) = 2\left|\angle KVN\right|.$

Whence $|\angle KVN| = \pi/2$, because of $\omega_1 + \omega_2 + \omega_3 + \omega_4 = \pi$.

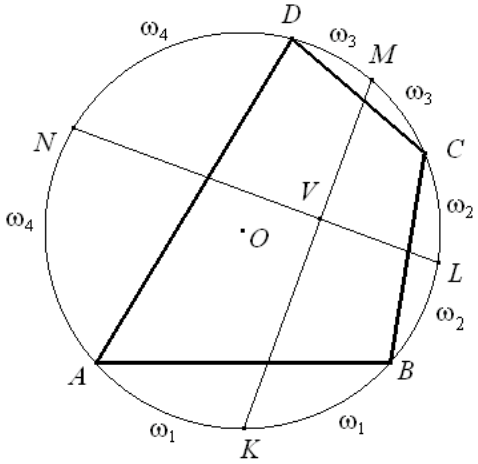

Figure 13

Let I is the incenter of given triangle ABC and $D \neq C$ is common point of circumcircle and bisector of angle ACB. Prove that $|DA| = |DI|$.

Solution. Without limiting the generality suppose that radius of circumcircle $r = 1$. Then the arc length is equal to the corresponding central angle. In accordance with the denoting in Figure 14 we get $2\varphi = \omega_1 + \omega_3 = 2\varepsilon$.

So triangle AID is isosceles, $|DA| = |DI|$.

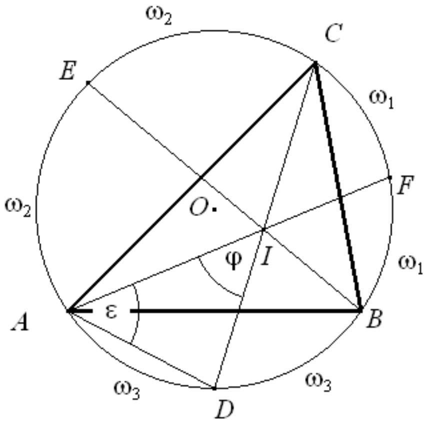

Figure 14

5 Conclusion

Application of dynamic geometry in schools requires new approaches to teaching. If it turns out that it is more efficient and rational, some traditional practices and curriculum units should be replaced by new;. Aim of this paper was to show one possibility.

I successfully used above-mentioned practices at work with mathematical gifted pupils on workshops (trainings for math. competitions). I believe that it allow better understanding of geometric properties of circles. The procedure from section 3 shows the inscribed angle theorem as a result of symmetric properties of circle. Also intersecting chords theorem is a direct result of the wonderful circle symmetry. It is a stronger tool than the inscribed angle theorem.

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