ENHANCING FUNCTIONAL THINKING USING THE COMPUTER FOR REPRESENTATIONAL TRANSFER

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The area of functional thinking is complex and has many facets. There are several studies that describe the specific difficulties of functional thinking. They show that the main difficulties are the transfer between the various representations of functions, e.g. graph, words, table, real situation or formula, and the dynamic view of functional dependencies (process concept of a function). Interactive Geometry Software allows the visualization of the dynamic aspect of functional dependencies simultaneously in different representations and offers the opportunity to experiment with them. The author presents and discusses the potential of two interactive learning activities that focus on the dynamic aspect of functional thinking in a special way. Some preliminary results from a first adoption of the activities in class are presented. Resulting research questions and plans for further research are stated.

Keywords: Functional thinking, representational transfer, Interactive Geometry Software, Interactive learning activity, empirical study.

THEORETICAL BACKGROUND

Functional Thinking – Concept and Relevance

In Germany the term 'functional thinking' was first used in the 'Meraner Reform' of 1905. The 'education to functional thinking' was a special task of the reform. Functional thinking was meant in a broad sense: As a common way to think which affects the whole mathematics education (Krüger 2000). In the 60s and 70s the impact of functional thinking in the above sense on the mathematics curriculum in Germany was very low. Since the 80s it regains importance although not in the broad sense of the Meraner Reform. A common definition of functional thinking derives from Vollrath (1989): 'Functional thinking is the typical way to think when working with functions'. Functional thinking in this sense is strongly connected to the concept of function. In the german mathematics curriculum the 'idea of functional dependency' is one of five central competencies, which form the mathematics education (Kultusministerkonferenz 2003).

The concept of function and functional thinking includes many aspects and competencies: On one hand functional dependencies can be described and detected in several representational systems like graphs, words, real situations, tables or formulas. On the other hand the nature of functional dependencies has different characteristics (Vollrath 1989 or Dubinsky, Harel 1992): Functional dependency as a pointwise relation (horizontal, static aspect), functional dependency as a dynamic process (aspect of covariation and change, vertical aspect), Functions viewed as objects or as a whole.

There are many studies (e.g. Janvier 1978, Müller-Philipp 1994, Swan 1985, Kerslake 1981) describing the following main difficulties and misconceptions concerning functional thinking:

The interpretation of functional dependencies in different representations and the representational transfer is a main difficulty. Especially the interpretation of functional dependencies in situations and the transfer to e.g. the graphical representation and vice versa causes problems. For example: graphs are often interpreted as photographical images of real situations ('graph-as-image misconception'), which is mainly caused by the inability to interpret the functional dependency dynamically. Especially distance-time graphs are often interpreted as movement in the plane.

The above difficulties were affirmed by written tests the author gave to either 10th class students and to university students who just started their study on mathematics. Based on the problems in the test the interactive learning activities, which we describe below, were built. Figure 1 shows one of the problems (Schlöglhofer 2000) from the tests.

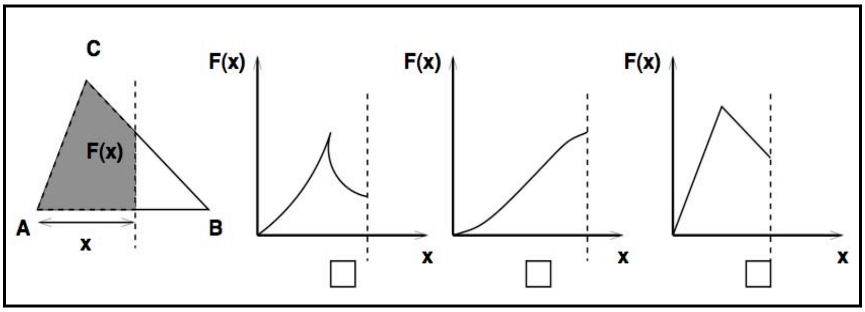

Fig. 1: The dashed line moves rightwards. F(x) is the area of the grey part of the triangle dependent on the distance x. Which graph fits and why?

Only 66% of about 100 university students made their cross at the graph in the middle. Giving the problem to sixteen 9th and 10th grade high school students, resulted in only 37% correct answers. The main mistake was to put a cross at the graph on the right side. The reason for this choice was usually given by a statement like: The area [of the graph on the right side] is just like the area F(x).

The chances of Interactive Geometry Software

When using the computer in classrooms on the topic functions one might think immediately of using Computer Algebra Systems (CAS). Most studies about the use of the computer when working with functions are about using CAS, e.g. Müller-Philipp (1994), Weigand (1999), Mayes (1994). While CAS is input/output based and gives back information and changes asynchronously, the use of Interactive Geometry Software (IGS) allows interactivity and gives immediate response. This difference will be used to emphasize the dynamic view of functional dependencies.

Especially the software Cinderella includes a functional programming language called *CindyScript*. This enables the teacher to create learning activities and own teaching material like the ones described below by using standard tools (Kortenkamp 2007).

DESIGN OF THE ACTIVITIES AND CONCEPTUAL BASIS

Main research question

The learning activities are designed with regard to the following research question:

Is it possible to enhance the dynamic aspect of functional thinking by dynamically visualizing functional dependencies simultaneously in different representations and by giving the opportunity to experiment with them?

General design ideas and concept

We developed two interactive learning activities (joint work with Andreas Fest). The activities consist of single Java applets embedded into a webpage and can be used without prior installation with a standard Internet browser. The applets are built with the IGS Cinderella and are accessible by using the links on the webpage http://www.math.tu-berlin.de/~hoffkamp.

Figure 2 shows the typical design of a learning activity. Next to the applet there is a short instruction on how to use the applet and some work orders. The students are asked to investigate and describe the functional dependency between the distance A-D and the dark (if coloured: blue) area within the triangle.

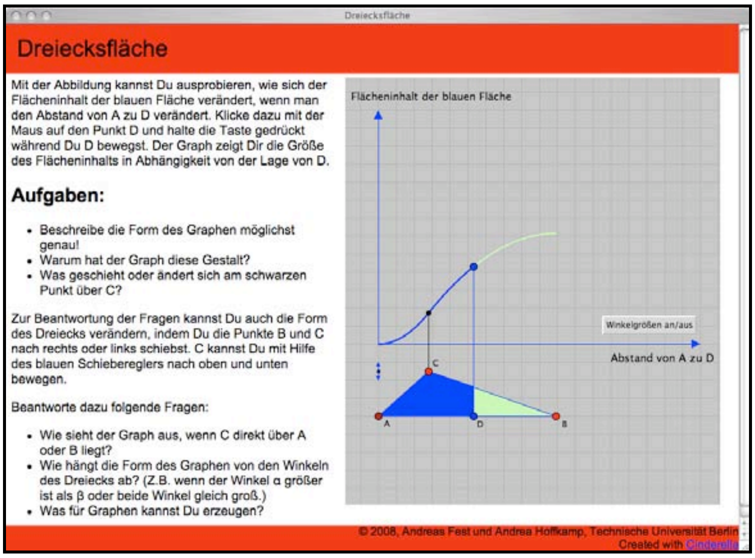

Fig. 2: Interactive activity 'Dreiecksfläche' ('Area of a triangle'). Moving D makes the dynamic aspect visual. Moving B and C changes the triangle and the function itself.

The learning activities have the following conceptual and theoretical ideas in common:

Connection situation-graph: The starting point is a figurative description of a functional dependency, which is simultaneously connected to a graphical representation. The graphical representation was chosen, because it relates to the covariation aspect in a very eminent way. As analysed by von Hofe (1995) students are able to establish 'Grundvorstellungen' (GV) more easily when an imaginable situation is given. GV's are mental models connecting mathematical concepts, reality and mental concepts of students. Rich GV's of the functional dependencies are necessary to succeed in problem solving processes.

Language as mediator: The students are asked to verbalise their observations in their own words. Janvier (1978) emphasizes the role of the language as a mediator between the representations of the functional dependency and the mental conceptions of the students.

Active processing assumption: According to the cognitive theory of multimedia learning of Mayer (2005) humans are actively engaged in cognitive processing in order to construct a coherent mental representation. The activities are conceptualized as attempt to assist students in their model-building efforts. Therefore the activities allow to experiment with different representations of the functional dependencies. At the same time the actions of the user are limited to focus on the dynamic view of the functional dependencies.

Two levels of variation: The activities allow two levels of variation. First, one can vary within the given situation. This visualizes the dynamic aspect. To understand a dynamic situation one needs to construct an 'executable' mental model to achieve mental simulation. The idea is to support the mental simulation processes visually (Supplantation, Salomon 1994). Secondly, one can change the situation itself and watch the effects on the graph. We will call this **meta-variation**. Meta-variation allows the user to investigate covariation in several scenarios. It is variation within the function that maps the situation to the graph of the underlying functional dependency and changes the functional dependency itself. This leads to a more global view of the dependency. Therefore meta-variation refers to the object view of the function. To understand the covariation aspect one needs to find correlations between different points of the graph in order to describe changes. This requires a global view of the graph. For example the property 'strict monotony' of a graph is a global property and therefore refers to the object view of a functional dependency. But to describe it in terms of 'if $x>y$ then $f(x)>f(y)$' one has to understand the covariation of different points of the graph.

Low-overhead technology and practicability: To work with the interactive activities there is no special knowledge of the technology necessary. The activities make use of the students' experience with Internet browsing (actions like dragging, using links, using buttons etc.). The students (and the teachers) can work directly on the problems without special knowledge of the software and the software's mathematical background. This is important especially with regard to time economy.

Learning activity 'Die Reise' ('The journey')

Based on the conceptual ideas above the learning activity 'Die Reise' ('The journey') was developed. Like the activity 'Dreiecksfläche' it is adapted from a problem (Swan

1985) the author gave to university students and 10th grade students within a written test. After using 'Die Reise' in classroom within a first study the activity was worked over. Some results of the study are presented below. The activity in its current version consists of three parts. Part one is about the transfer situation-graph (Fig. 3): A car advances from Neubrandenburg (top of the map) to Cottbus (bottom of the map). The graph shows the corresponding distance-time graph for the journey.

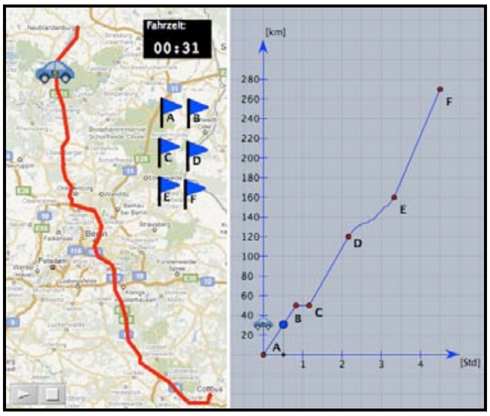

Fig. 3: Applet within the first part of 'Die Reise'. The point on the distance-time-graph is movable. The students are asked to mark the positions A-F with the flags on the map.

Part two of 'Die Reise' (without figure) refers to the first level of variation (visualization of the dynamic aspect in the given situation). It shows the distance-time graph of part one again together with the corresponding velocity-time graph. The work orders aim at interpreting the slopes in the distance-time graph in connection with the velocity-time graph.

Part three refers to the level of meta-variation (figure 4).

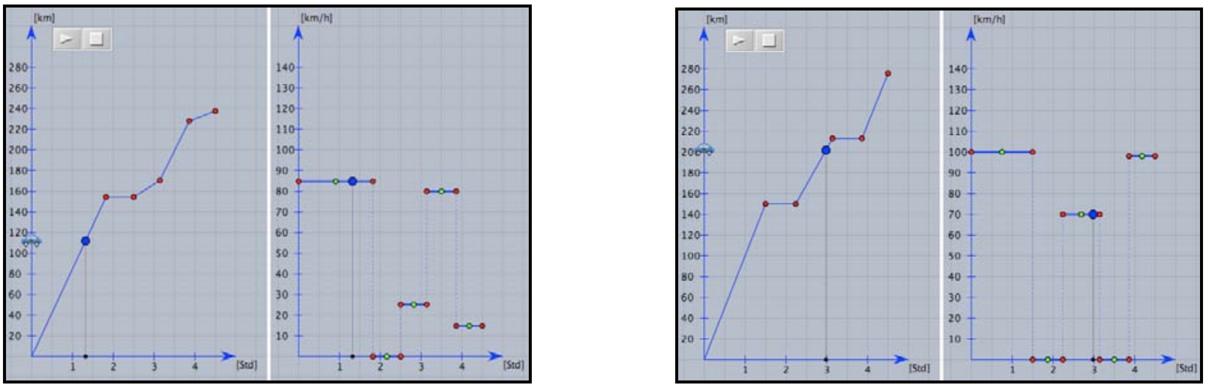

Fig. 4: Meta-variation in 'Die Reise'. Besides moving the points on both graphs, the bars in the velocity-time graph can be moved vertically and the width of the bars can be changed.

FIRST STUDIES

Setting and methods

The activities 'Dreiecksfläche' and 'Die Reise' were tested with 19 respectively 32 secondary school students of age 15-16 (10^{th} class) in a block period of 2x45 minutes in each case. The students were not prepared to either the topic or the special use of technology. A worksheet was prepared which contained the Internet address of the interactive activity and some questions to work on. The students had to start on their own using the instructions of the worksheet. To provoke discussion and first reflection about the problems two or three students worked together. Afterwards the solutions were discussed in class. The results of the studies are based on student observations during their work with the computer, general impression of the discussion in the class, a short written test and a questionnaire. In addition the computer actions and student interactions of one student pair was recorded while working with the activity 'Die Reise'. All teaching material, tests and questionnaires can be found on www.math.tu-berlin.de/~hoffkamp.

The studies were conceived as preliminary studies with the following aims: Test the interactive activities and work them over for further studies, specify further research questions, create a study design for a larger study based on the experiences made.

Results and discussion

Computer-aided work and work with the activities in general:

The concept of low-overhead technology and practicability was successful. The students were able to work with the interactive activities without further instruction. This is also important concerning time economy, especially from the teachers' point of view.

The use of the computer had a very positive effect on the students' motivation. This is caused by many factors. For example the students appreciated to work autonomously in their own tempo following their own train of thoughts. They also highly appreciated that the computer takes over annoying actions like drawing or calculating. This is a crucial point especially for slow-writers and was observed when watching a recorded sequence of the students' working phase. The sequence shows that the order 'Draw a suitable distance-time graph' really blockaded and frustrated the student. The following student statements from the questionnaire confirm the above comments:

> Question: Is there something special you like when working with the computer?
>
> Answer 1: It is less monotonous and the lesson is organized differently. You learn by means of a different learning aid, which allows a better imagination. The studious atmosphere is more comfortable. You do not have to follow the group's train of thoughts.
>
> Answer 2: The computer makes the calculations and I do not have to write so much, which means that it cannot be smeared and illegible.

> Answer 3: That I can work independently (without teacher). One can use his own mistakes to come to the right result.

Statements like answer 3 were made several times. The students had the impression that they were able to use their mistakes in a productive way. Moreover the computer-aided work allowed for a better internal differentiation of the learner group. Slowly learning students asked the teacher for help more often than more advanced students, but they still worked independently for longer periods.

Effects on functional thinking:

By visualizing the representational transfer dynamically the students were forced to focus on the dynamic aspect of functional thinking and they seem to have established (more or less) adequate mental models integrating the dynamic view. Many student answers on the questionnaire aim at the aspect of 'dynamic visualization':

> Question: What is different for you when you use the computer to work on mathematical problems?
>
> Answer: Because of the visualization I am able to watch the problem from different perspectives and this makes it easier to solve it.
>
> Question: Can you say what exactly you understood better by using the computer?
>
> Answer 1: That I could **see** the problem.
>
> Answer 2: How the graph changes when changing the triangle.
>
> Answer 3: I liked this form of figurative illustration that was given directly when changes were made because it is easier to understand something by watching it.
>
> Answer 4: The motion. When one graph moves although you use the other graph.

The second question was used to find out what aspects of the activities where considered by the students as showing them something new. In this sense many student answers aim at the level of meta-variation. As explicated above the level of meta-variation is connected with the object view of a function, a view, which is not (fully) attained in the age group the author is looking at (Sfard 1991). The student answers lead to the assumption that meta-variation makes the object view accessible for cognitive processes (in a may be implicit way) and could be a step towards the perception of a function as an object. This assumption is strengthened by the results of the tests. Figure 5 and 6 show some results from the written test.

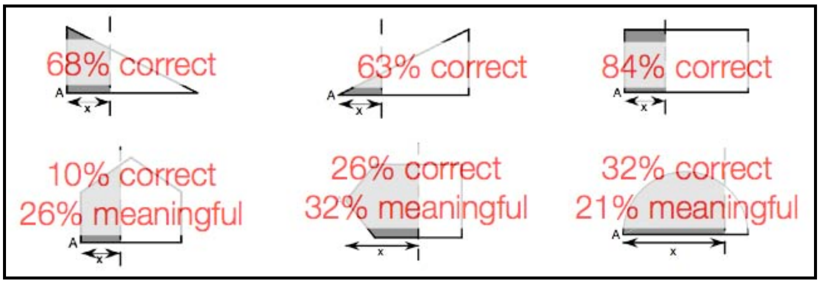

Fig. 5: The students had to sketch graphs describing the dependency between x and the grey area F(x). The figure shows percentages of correct and meaningful graph sketches. An answer was 'meaningful' when the graph was strictly increasing, but e.g. left and right turn were mixed up.

As seen from figure 5 the students by majority seemed to have created an 'idea' of the dynamics of the functional dependency as far as the solution of problems like the one above is concerned, although it was still difficult to adapt the concept to other situations (here: other forms in line two of figure 5). However the students got aware that changes, variations, certain points (e.g. inflection points) and properties (e.g. symmetry, monotony) have a graphical correspondent, which gives qualitative information about the functional dependency.

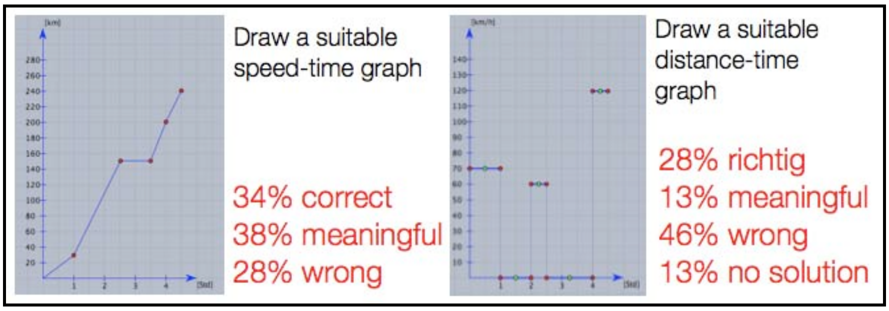

Fig. 6: The students had to draw suitable graphs to the given graphs above. Graphs were 'meaningful' when the graphs had 'correct shapes', but some slopes where done wrong.

Figure 6 shows some results of the post-test within 'Die Reise'. Most of the students were able to draw suitable speed-time graphs to given distance-time graphs. The other way round – from speed-time graphs to distance-time graphs – was more difficult. The results confirm the assumption that the potential of meta-variation in order to enhance the understanding of the dynamic aspect of the functional dependencies seems to be high. Furthermore the activities seem to allow an easy qualitative approach to concepts of calculus. In case of 'Die Reise' the applets visualize the physical intuition of the fundamental theorem of calculus. When discussing the question 'Can you see from the speed-time graph how far the journey is?' in class, the students finally ended up with an intuitive concept of integration.

The results from the preliminary studies lead to the following hypothesis: *Although the object view is more advanced, it facilitates the understanding of the covariation aspect and the establishment of mental models with regard to the dynamic view of functional dependencies.*

The class discussion of the results – which mainly consist of verbalisations of the properties of the functional dependency – ran pretty smooth. The students were highly engaged in making contributions to the discussion. But it was obvious that there was a high need for reflection of the students' train of thoughts since the student answers were mostly superficial. Concerning 'Die Reise' some test results showed, that the 'graph-as-image misconception' is very persistent in the sense of interpreting distance-time graphs as movement in the plane. Based on these experiences the first two parts of the learning activity 'Die Reise' were modified to their current version.

OUTLOOK

The preliminary results of the first studies give valuable hints for the direction of further research. Basing on the conceptual ideas described above a third interactive activity will be developed and pretested. It is planned to conduct a larger qualitative study using the three activities. The leading question is how the work with functions within the activities affects functional thinking itself. The level of meta-variation is a central idea. It leads to the concepts of calculus and may be used as a qualitative approach to school-analysis in the context of propaedeutics.

The following research questions are of interest and will guide our future research:

Main question: Is it possible to enhance the dynamic aspect of functional thinking by dynamically visualizing functional dependencies simultaneously in different representations and by giving the opportunity to experiment with them?

Further questions:

- Do the students establish GVs concerning the dynamic view of functional dependencies? What sort of GVs do the students establish?
- Which elements of the applets have a positive effect on the dynamic view of functional dependencies?
- Is it possible to distinguish types of students who get along better or worse with the learning units?
- How do slow learners deal with the units compared to more advanced students?
- How can we use computer-based activities like these as diagnostic tools?

REFERENCES

Dubinsky, E.; Harel, G. (1992). The Nature of the Process Concept of Function. In: Dubinsky/Harel (eds.): The Concept of Function, MAA Notes 25, pp. 85-106.

Kerslake, D. (1981). Graphs. In: K. Hart (Ed.) Children's Understanding of Mathematics: 11-16. pp. 120-136. John Murray, London.

Janvier, C. (1978). The Interpretation of Complex Cartesian Graphs Representing Situations. PhD thesis, University of Nottingham, Shell Centre for Mathematical Education, Nottingham.

Kortenkamp, U. (2007). Combining CAS and DGS – Towards Algorithmic Thinking. In: *Symbolic Computation and Education.* Eds.: Shangzhi, Li et al., World Scientific.

Krüger, K. (2000). Kinematisch-funktionales Denken als Ziel des höheren Mathematikunterrichts – das Scheitern der Meraner Reform. *Mathematische Semesterberichte,47*, pp. 221-241.

Kultusministerkonferenz (2003). Bildungsstandards im Fach Mathematik für den Mittleren Schulabschluss. Luchterhand, Darmstadt.

Mayer, R. (2005). Cognitive Theory of Multimedia Learning. In: *The Cambridge Handbook of Multimedia Learning*, Ed.: Mayer, R., Cambridge University Press, New York.

Mayes, R. L. (1994). Implications of Research on CAS in College Algebra. *International Derive Journal*, 1(2), pp. 21-37.

Müller-Philipp, S. (1994). Der Funktionsbegriff im Mathematikunterricht. Waxmann Verlag GmbH, Münster/New York.

Salomon, G. (1994). Interaction of media, cognition, and learning. Hillsdale, NJ: Lawrence Erlbaum Associates.

Schlöglhofer, F. (2000). Vom Foto-Graph zum Funktions-Graph. *mathematik lehren*, 103, pp. 16-17.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational studies in mathematics*, *22*, pp. 1-36.

Swan, Malcolm et al. (1985). The Language of Functions and Graphs. Shell Centre & Joint Matriculation Board, Nottingham.

Vollrath, H.-J. (1989). Funktionales Denken. *Journal für Mathematikdidaktik, (10)1*, pp. 3-37

Vom Hofe, R. (1995). Grundvorstellungen mathematischer Inhalte. Spektrum Akademischer Verlag.

Weigand, H.-G. (1999). Eine explorative Studie zum computergestützten Arbeiten mit Funktionen. *Journal für Mathematikdidaktik*, 20(1), pp. 28-54.