UNPACKING CHILDREN'S ANGLE "GRUNDVORSTELLUNGEN": THE CASE OF DISTANCE "GRUNDVORSTELLUNG" OF 1° ANGLE

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Results of the last thirty years in mathematics education have shown the importance of an operational concept development. One of the geometrical concepts that have been researched for years already, however, not with the particular focus on its systematic teaching in school mathematics, is the concept of angle. In this paper we focus on children's understanding of the angular size of 1° and its development obtained through a test followed by a task-based interview. The interview results with 9 pupils showed that they have a fragmented understanding of the angle concept, enabling them to fully grasp what 1° angle is. Moreover, many of the children's misconceptions were directly connected to the measuring tool, namely set square, and angle notation. Implications for systematic teaching of the angle concept are given at the end.

INTRODUCTION

The angle concept is a fundamental concept of plane geometry and central to the development of geometric knowledge and thinking. This concept is not only relevant for the entire geometry teaching, but also in everyday situations and in different careers. In Germany, the angles, namely right angle, get introduced at the elementary level, but its systematic learning starts at 5th grade and lasts throughout grade 10. Both the state curriculum and the standards give guidelines as to what ideas, knowledge with respect to the angle concept should be learned. The angle concept, although being an elementary concept of plane geometry, poses problems for many middle-school and high-school students; the students have no sense of angle size, have fragmented knowledge of angle aspects, lack knowledge of angle attributes, do not understand the protractor as a measuring tool, and so on (Dohrmann & Kuzle, 2013, in press; Krainer & Cooper, 1990; Mitchelmore & White 1995, 2000; Van de Walle, 2001). For that reason, project WiKUL (Winkel konstruktiv unterrichten und lernen, that is teaching and learning of angle concept under constructivist epistemology) was developed. The goal of the project is to understand which of these ideas, and operations about angle are encountered by middle- and high-school students and how the angle concept can be conveyed to students in a meaningful manner and by using fundamental ideas of concept learning to prevent development of angle misconceptions. For the purpose of this paper we focus on one of these aspects, namely students' angle measure understandings past elementary level, which was focus of previous research (Kaur, 2013; Mitchelmore & White, 1995, 2000) and have prevailed in our previous research (Dohrmann & Kuzle, 2013, in press).

THEORETICAL PERSPECTIVE

The nature of the angle concept has been vividly debated for over two thousand years, and the discussion is not close to be over (Krainer & Cooper, 1990). This discussion resulted in three different definitions or aspects as well as different representations being typified in school mathematics. Having this complexity in mind, research has shown that students have serious misconceptions about the concept of angle based on their personal experiences. For that reason a somewhat radical approach is needed to alter preexisting concept structure. With this in mind, the conceptual change theory is becoming more prominent in the mathematics education research to explain student's difficulties in learning mathematical concepts (Posner et al., 1982). According to this this theory, that draws from both Kuhn's sociology of science and Piaget's developmental psychology, learning can occur in two manners: (1) new knowledge is added to the prior knowledge (assimilation) and (2) old knowledge is first reconstructed as a result of disequilibrium or conflict when confronted with new knowledge (accommodation) before the conflict can be resolved or it gets overthrown (rejection) by the learner. Following this process students can then undergo the process of accepting, integrating and using the new concepts.

Though the conceptual change approach has been proven to be a fruitful framework for analyzing student difficulties, it does not exhaustively reflect the complexity of the learning process, student's understanding of a particular concept nor student's learning difficulties. Students due to the fact that mathematical concepts and symbols, which are used in the teaching of mathematics, are often understand essential reasons for these problems with a totally different meaning from what was intended by the teacher (vom Hofe, 1998). For that reason, different concepts of the generation of „mental models" have been developed to counteract these problems. In Germany these mental models, which bear the meaning of mathematical concepts or procedures are called *Grundvorstellungen* (GVs), which emphasize the constitution of meaning as a central aim of mathematical teaching. They can be interpreted as "elements of connection or as objects of transition between the world of mathematics and the individual world of thinking" (vom Hofe, 1998, p. 320), which show structural and functional aspects of a mathematical subject. GVs are not static mental models, which are valid forever, but its generation is a dynamic process of changes, reinterpretations and modifications as involvement with new mathematical subjects takes place. It is a cognitive net in which single GVs are in correlation to others.

GVs cannot be directly studied but require the need to be aware of three different types of behavior, prescriptive (basic idea), descriptive (individual image) and constructive. In mathematical literature, prescriptive notion of angle GVs are given describing adequate interpretations of the core of the respective mathematical contents which are intended by the teacher in order to combine the level of formal calculating with corresponding real live situations. For instance, from a normative aspect (basic ideas) of 1° angle can be the amount of openness between the two rays of an angle, which corresponds to 360th part of the circle circumference with *degree* as a unit of measure equal to it, which openness is so small (on paper) that one can barely see the difference between the too rays. However, descriptive notion focuses on describing ideas and images, which students actually have and which usually more or less differ from the relevant mathematical thoughts intended by mathematical instruction. Thus, in the teaching-learning context it is important that the teacher specifies an adequate basic idea of 1° angle, so that the students do not generate an image detached from it, such as 1° being understood as a Euclidean distance between the two rays or as measure of the extent of a two-dimensional shape. The third perspective focuses on developing and confronting students with learning situations that would allow them to change, rebuild, and refine their individual images.

In summary, when thinking about the teaching-learning process, the first focuses on ideas that has to be formatted by the students, the second on images, which have been activated by a student and third initiated by the teacher as a result of faulty or not fully developed basic ideas. In this paper we focus on the process of teaching-learning of the angle measure of 1° with the interplay between individual images and basic ideas, and how these can lead to the constitution of basic ideas of the students in a psychological sense.

METHODOLOGY

The study took place in a Montessori comprehensive school in the state of Saxony. We administered a WiKUL test to approximately 300 students in grade 5 to grade 10. The purpose of the test was to grasp and understand their existing ideas and aspects about the angle concept, and to obtain an image for the understanding of the concept and the associated operations. The students had 45 minutes for the test. The test items were aligned with the Saxon curriculum and consisted of two types of items, that focused on the following two aspects: (a) intra-mathematical knowledge on both grade and across-grade tasks, and (b) patterns of thinking in application tasks about the angle concept (Dohrmann & Kuzle, in press). A special test item was used, namely Anna-letter developed by Thomas Jahnke, as a source of data for the pupils' individual images about the angular size of 1°. In this data source a 12-years old bright girl by the name of Anna is introduced asking students for an advice or help. For the purpose of our study, Anna-letter focused on asking the pupils to help Anna understand what 1° angle is:

> Dear …,
> Yesterday we repeated angles in math class. Our teacher wanted to know what 1° is. With the question I was totally overwhelmed. Although I know that we have constantly used this, I cannot exactly explain what 1° means. Can you please help me? Maybe you can also draw a sketch.
> Thank you and best regards, Anna.

This item was used for 5-10th graders. By using this data source and through children's communication, representations and arguments, we obtained an insight into children's images about the 1° angle. The analysis of Anna–letters occurred in several steps where both inductive and deductive methods were used as suggested by Patton (2002). The analysis showed that pupils held many different images about 1° angle (Dohrmann & Kuzle, in press) with distance GV about the 1° angle being highly coded and across different grades (ca. 10% of children). This GV was assigned when word distance was used in the verbal explanation, and/or when 1° was equated with a distance measure (e.g., 1°=1mm, 1° equals distance between two dashes on the set square).

To confirm written explanation and to better understand this GV, nine pupils were chosen on the basis of their contrasting responses (GVs and misconceptions) about the angular size of 1° and interviewed; two from grades 5 and 6, one from grades 7, 8 and 9, and two from grade 10. The interviews lasted ca. 15-20 minutes and focused on student's elaboration of Anna-letter and how this GV developed. In addition, another instrument was used, namely Anna-video. In it a girl Anna measures the angle as described by each pupil in the Anna-letter; she measured the angle by measuring the distance between the two rays, concluding that since the distance between the two rays equals 1mm, the angular size corresponded to 1°.

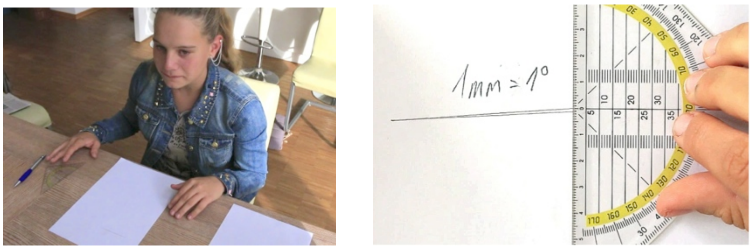

Figure 1: Anna-video.

The children were supposed to comment on Anna's solution and give us a better understanding of their image by explaining their notion, refining, rejecting or rethinking their distance GV. In other words, we were interested how the children deal when confronted with new experiences and challenges as explainer earlier.

This data was again analyzed using content with contrasting comparative methods (Patton, 2002). To increase the reliability of the study, both authors coded the data separately and meet to discuss the codes. When agreement was meet, the code was assigned.

CHILDRENS' DISTANCE GRUNDVORSTELLUNGEN OF 1° ANGLE

A summary of findings is presented in terms of children's distance image(s) in angle context, their understanding of 1° angle given through elaboration and arguments on the basis of their Anna-letter and reaction to Anna-video, and relationship among their identification of 1° angle and its representation on paper and set square. Brief descriptions are provided for the categories with quotes from participants.

Individual image of "distance" in angle context – in mathematics, from a normative perspective distance is a function that describes how apart objects are. In the angle context the normative aspect of distance is described as the length of the unit circle arc enclosed by a particular angle. However, the descriptive perspective exhibited by the participants was different. They held three different images of distance GV with respect to 1° angle: (1) the distance between the two dashes on the set square which was equal to 1mm (N=3), (2) the 1 mm distance between the two half-rays (N=3), and (3) the length of the arc closed by the 1° (N=3). First image was observed by Lynn (5th grade), Elli (5th grade), and Toni (7th grade). These children showed the 1° angle on the set square; it was identified as a plane between zero and two dashes on the set square. However, the distance between the dashes was then estimated to 1mm and equated with the 1° angle.

Lynn: Well, I just thought that 1mm …So when I have here the set square, that here between the two lines maybe 1mm is …

Toni (7th grade) argued similarly, but by using the half-circle scale.

Toni: So, I'd say that for instance here on the set square 1° is 1mm here on the circular edge marking.

Joanna (6th grade), Toni (7th grade), and Mike (8th) similarly identified 1° degree as 1mm "distance" but between the two half-rays, whereas Elaine (10th grade) as 1cm distance. Ally (6th grade), Jess (9th grade) and Layla (10th grade) associated the "distance" with the arc length. For instance, Jess viewed it as 360th part of circle, which had the length of 1mm. Based on the sample we can conclude that independent of the grade level, pupils held these different misconceptions about the 1° angle. Hence, these different sub-misconceptions were stabile throughout grades 5 to 10. Moreover, the source for some was traced to the tool itself, namely the set square, used in all grades when teaching and learning the angle concepts.

Individual image of 1° angle on the basis of Anna-instruments given through elaboration and arguments – children differently reacted to Anna-video. Four pupils tried to make sense of the newly acquired information by trying to make connections to their own learning. At the end they assigned it to another way of measuring an angle. Hence, they accommodated the new technique into their existing scheme.

Elaine: I'm not sure, maybe one can measure an angle like that. I think, when she would have measured the angle a bit further, then it would have become bigger, the distance… However, I guess that it does not make a difference how much one extends the dashes.

Five pupils after trying to make sense of it, consciously rejected the new technique.

Jess: I would tell her [Anna], that one has to differently lay the set square. So one lays it onto the angle where zero is und that one reads it off like that, but yeah I write it myself incorrectly.

Interviewer: At what point would you say that you wrote it down incorrectly?

Jess: Well I assumed the same ideas… I also measured it from the top and the said to myself '1mm is also 1°'. But when I think about it, it is clear to me, that that is not correct and that I also incorrectly lied down the set square.

As shown in this one excerpt, as a consequence they got aware how their explanations using mathematically inappropriate language led Anna to false actions. These few excerpts show that pupils generated multiple related ideas for this situation. In other words, conceptual change was exhibited allowing children to challenge their written and verbal explanation about the 1° concept. These were exhibited on two meta-levels: language and situation. In the former certain words and formulations were used to describe the distance GV that could be interpreted in different manner. However, the children used the word "distance" in a non-linear context, as they could not find another word for it, such as "angle openness". In the latter, Anna-letter and Anna-video were seen as two independent entities. The ideas from the video were regarded as a new concept that got either accommodated or rejected.

Relationships among identification of 1° angle and its representation – the pupils identified 1° angle on the set square and represented in then on paper. Collectively, the pupils identified 1° angle either along the leg of the set square or on the half-circle scale. More precisely, four pupils identified it as an angle between first two dashes on the leg, whereas only two pupils as an angle between any to dashes lying next to each other on the leg of the set square. Three students referred to the half-circle scale; similar to the above description, two pupils identified it as an angle between any two dashes next to each on the half-circle scale. Surprisingly, one student, Joanna (6th grade), claimed 1° angle not existing on the set square.

Interviewer: And now show me a 1° angle on it [set square].

Joanna: That doesn't work since it [scale] begins with 10°. So, it's a bit difficult to measure.

Interviewer: One cannot see 1° angle on the set square?

Joanna: Nah-ah … This is merely 170°, when one doesn't have an entire half-circle.

Pupils were then asked to draw 1° angle using the set square or the protractor. Collectively almost all pupils identified 1° angle as a part of plane bounden by two half-rays and an arc. That is, as a part of plane bounded by the arc and close to the vertex; what was "outside" of the arc was not identified as 1° angle.

Elaine: That's 1°.

Interviewer: Show it one more time.

Elaine: Here and here, so here behind would be 1°…Well, maybe here would also be 1°, because when one … so I have here… nah, in fact it has to be here in the front.

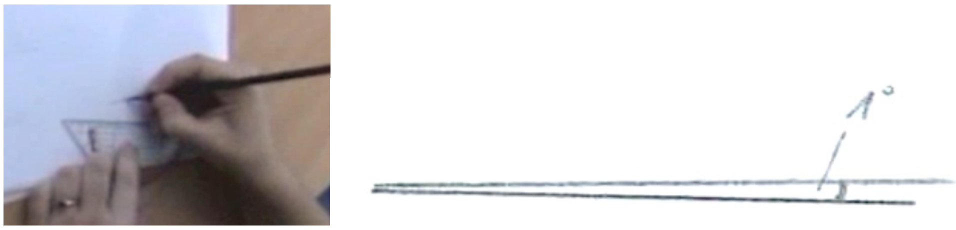

Figure 2: Elaine's identification of 1° angle through fixation on angle notation.

Hence, 1° angle was identified through notation. Such fixation on notation enabled the conceptual understanding of the angle concept; she thought about extending the rays, but then concluded that 1mm would not come out. At the end 1° angle was again equated to 1mm, where 1mm was the Euclidean distance between the two rays and part of the angle enclosed the by vertex and plane bounded by the 1mm length. Similar behavior was observed with other participants. Thus, interplay between both ideas about 1° angle and its notation allowed for developing deep misconception not only about 1° angle, but angle concept and its main ideas.

CONCLUSIONS

The results presented here show that students have a fragmented understanding of the angle concept as shown in previous studies. However, we have shown that the development from grade 5 to grade 10 continues to cause a rising gap between the angle concept and its main ideas. In particular as a consequence pupils cannot fully grasp what 1° angle is, cannot fully describe it or show it, and reject fallible actions of others. Different misconceptions, such as notation fixation, can severely inhibit identification or construction of adequate images. Secondly, the results have shown that some children do have the mathematical understanding of 1° angle, but missing appropriate mathematical language competencies to communicate their thinking. Thirdly, many of children's misconceptions were directly connected to the set square. Through the routine tasks procedures were learned and practiced, without building a deeper understanding of the tool and its affordances, which then inhibited conceptual understanding of the angle concept and its operations. Moreover, the tool emphasizes the static angle perspective, but is not suitable for developing the dynamic angle perspective.

The pupils cannot assimilate the ideas given by the teacher, nor can these be transferred into the pupil's mind. Children construct their own images, ideas, and models based on the available teaching-learning situations. As a result of this and other research (e.g., Dohrmann & Kuzle, 2013; Kaur, 2013; Mitchelmore & White, 1995, 2000), we advocate for teaching and learning concept oriented towards the student, on the understanding and the application with respect to the angle concept on the basis of a GV-grounded access and by using a didactically more appropriate angle tool. In more details, the process of teaching and learning focused on transposing basic ideas on the one hand, and being sensible for the individual images on the other hand, are didactical means for inviting students developing adequate meaning of mathematical concepts. We are currently developing materials for teachers and students with these ideas in mind. On the one side, the materials should allow students with the situation that support discovery of fundamental ideas and aspects in order to develop understanding for corresponding operations. Moreover, the students need to development the ability to handle the daily situations for which the angle concept is crucial. Last but not least, we want to support teachers by developing materials that would allow them to analyze children's strategies and mistakes, and hinder misconceptions to enable the further progress. Such appropriate teaching-learning situation would allow for changes, reinterpretations, or modifications to basic ideas contributing to a greater understanding of a multi-faceted angle concept.

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