Development of Geometric Competencies –
Children's Conception of Geometric Shapes in England and Germany

Andrea Simone Maier
Christiane Benz
University of Education, Karlsruhe

Abstract

To investigate the geometric competencies of children from 4 to 6 years old in England, with a curricular pre-school system and Germany, with a more play-oriented kindergarten education, 80 children were given geometric tasks via clinical interviews. In this paper, some selected results are presented with its focus on children's conceptualisation of geometric shapes, which also becomes visible in their drawings. Furthermore, hypotheses with respect to possible influences of the single elementary concepts are formulated.

Introduction

Since international comparative studies like TIMMS and PISA, the topic of the early education and, with it, also the topic of early mathematics education and how education should look like, has been widely discussed. Research suggests that early learning is important in order to offer a basic education for all children.

Still, the remaining question is how education for 4- to 6-year old children should be designed. In Germany alone, there are a number of approaches and concepts as well as new emerging programs for pre-school education. However, it is not institutional or rather uniformly clarified if and how the unplanned, purpose free playing or learning through playing or constructivist learning (Schäfer, G. E.,2010, 2011; Rigall, A. & Sharpe, C., 2008; Puhani & Weber, 2005) should be replaced by systematic, curriculum based learning or instructional learning (Duncker, 2010; Preiß, 2006, 2007; Krajewski et al., 2007). On the one hand, there is a demand for protection from schoolification especially for younger children, but on the other hand it is important to support mathematical competences before entering school because we know that they are predictors for later success in mathematics (Schneider 2008, Dornheim, 2008) and to avoid existing learning capacities in children being exhausted.

With this in mind, the study at hand investigates the geometric competencies of children from two countries with different concepts of elementary education: Germany, where learning through play at present is the main concept for kindergarten education and England, where the elementary education is rather systematic and curriculum based and where the competencies of the children are tested via "stepping stones" which they should have acquired. There, the children enter school in the year when they have their fifth birthday, but many children go to a reception class before that. So the entering school age is about two years earlier than for children in Germany.

The topic of geometry was chosen because there have been less studies in this area than, for example, in number and counting, but it is still a very important aspect of mathematics as is illustrated in the following quote:

> *"No mathematical subject is more relevant than geometry. It lies at the heart of physics, chemistry, biology, geology, and geography, art and architecture. It also lies at the heart of mathematics, though through much of the 20th century the centrality of geometry was obscured by fashionable abstraction" (Sarama & Clements, 2009, p. 201).*

Theoretical Background

The focus of this paper is on the development of geometric concepts. First, it will be illustrated, what constitutes a concept, before two general theoretical models concerning concept development are presented. That following, some empirical results concerning the development of geometric concepts are shown.

Constitution of a concept

Franke (2007) defines a concept as follows:

„We speak of a concept , if it not only represents one single object – or incidence and so on – is meant, but a category or a class is associated with it, in which the concrete object can be classified" (Franke 2007, p. 72).

According to Franke (2007) a comprehensive conception of geometric shapes, as a concept for objects, is shown through being able to:

- name the shapes
- give a definition of the shapes
- show further examples of this category
- name all properties

There are different suggestions how such a comprehensive conception develops, one is illustrated in the following section.

Conceptualisation theories

Szagun (2008) proposes two theoretical approaches that illustrate how a concept develops. In the "semantic feature hypothesis" ("semantische Merkmalshypothese") general features are learned before specific features. For example, the child has learnt the word "dog", which is connected with one semantic feature and that is “four-legged”. Accordingly, the child would first call every four-legged animal (horse, cat, mouse…) a dog. With the time other semantic features, such as “barking” are added so that the word “dog” could be distinguished from “cow” for example. The features are either present or not and apply for every member of the class, e.g. “all kinds of dogs belonging to the category “dog” are four-legged and bark”. In contrast, in the “prototype theory” (“Prototypentheorie”), which is considered as the psychological more real theory, some members of a category are categorised as more typical than others (Szagun 2008, p. 134). For example a sparrow is a more typical bird than a chicken, although both belong to the subordinate concept “bird”. In addition to that, not every member of the category “bird” has the same features. Members having a lot of features in common are “prototype members” of the category bird (e.g. sparrow, robin) and members having fewer features in common are periphery members of the category bird (e.g. chicken). However, in order to give a complete picture of what we know of the geometric concept formation, how a concept develops has to be complemented by research findings on geometric concepts.

Empirical Background

With the observations of Piaget & Inhelder (1975a, 1975b), research focusing on children’s concepts of space and geometric shapes began. His research findings revealed that children younger than 4 years of age are not able to distinguish a circle, a square and a triangle, but consider all of these shapes as “closed” figures. With the age of 4, the children start to distinguish between curved and straight shapes but not among these classes: for example a circle is not distinguished from an oval and a square is not distinguished from a rectangle or even a triangle. At the age of 6, the children are able to name and to distinguish between geometric shapes. Since Piagets studies, several researches have either verified (Laurendau and Pinard 1970) or contradicted (Darke, 1982; Lehrer et al., 1998) some or all of the original hypotheses of Piaget (c.f. Hannibal & Clements 2008). Some studies reported for example, that even at an earlier age children were able to distinguish between curvilinear and rectilinear shapes (Lovell, 1959; Page, 1959). Another body of research has focused on children’s reasoning about geometric concepts that they have formed (van Hiele & van Hiele, 1986). The van Hieles, who also created a hierarchical developmental description, constitute that on the first level (pre-recognition) before the age of 4, children are not able to capture all of a geometric shape, instead only parts of the shape can be comprehended and properties can’t be explicitly realized yet. At the end of this level, children can distinguish between curvilinear and rectilinear shapes but not among these groups (in concordance with Piaget). On the next level, the visual level, up to 7 years, shapes are realized as whole entities. The following level,

the analytic or descriptive level, is representative for primary school children and goes up to the age of 9/ 10. The shapes are now distinguished by their properties. Correlations between different classes, e.g. squares and rectangles, can't be made yet. The other levels concern secondary school and beyond (University level).
Following these developmental models, there were several studies to proof the existence of such levels or the characteristic of such levels (e.g. Burger & Shaughnessy, 1986; Gutiérrez et al. 1991; Clements & Battista, 1992; Lehrer et al., 1998; Battista, 2007). As common ground, most empirical research confirmed that such levels exist and that they are useful in describing childrens' geometric concept development but that they are not discrete or independent. Moreover, it is difficult to relate a student to one single level for students were on different levels for different concepts and exhibited different preferred levels on different tasks (Burger & Shaughnessy, 1986; Battista, 2007). Thus, the assignment to levels does not seem to be strictly related to age or theme and with this, the hierarchical order of the levels is shaken. Other research proposes that the characteristics of the single levels develop at the same time but in diverse intensity (Clements & Battista, 1992; Lehrer, 1998).

Apart from this, there have been studies with the single focus on the development of geometric concepts in children (Clements & Battista, 1989; Clements et al., 1999; Hannibal & Clements, 2008), instead of investigating geometric competencies as a whole. There also have been studies what visual prototypes and ideas preschool children form about common shapes. Focusing on a few detailed empirical results, Clements et al. (1999) found that children identified circles quite accurately and had some difficulties in selecting squares, for they were less accurate in classifying squares without horizontal sides (Clements, 2004, p. 269f.). They had most difficulties in recognizing triangles and rectangles. The study revealed that children's prototype of a triangle seems to be an isosceles triangle and their prototype of a rectangle seems to be a four-sided figure with two long and two short sides and "close-to" square corners. Square prototypes only occur concerning position and there are no circle prototypes, for they all, except from size, look the same.

Although there have been several studies on the development of geometric concepts in children, there hardly have been any studies yet at this topic regarding different educational settings. The research at hand is a descriptive study to illustrate the understanding of geometric shapes English and German children in the age of 4 to 6 have and how these competencies develop in the course of one year. Furthermore, it was examined whether the children of this study could be grouped into a hierarchical stage model or rather into a dynamic developmental model.

Empirical Study

Research Questions

The underlying research questions are

(1) How do children solve the tasks to geometric conceptualization of shapes and how do they explain their proceeding?
(2) What differences in the development can be described after a year?
(3) Does the educational setting, the way how early learning is enhanced, influence the competencies of the children? And if so, how far?

Subjects

The research gathered 81 children, of which 34 are of English nationality and were attending a local primary school, near Winchester. The age of the children at this primary school ranges from 4 to 11 years. The other 47 children were from Germany and attending a kindergarten in Karlsruhe, where children from the age of three up to primary school can go.

Method

The study was conducted in the form of clinical interviews, of which the origins coincide with Piaget's early investigations into children's thinking (Ginsburg and Opper, 1998). The order of the tasks as well as the material was predetermined but in accordance with the nature of clinical interviews this order could be altered or complemented if some of the child's answers happened to be interesting or leading into another direction worth being examined. There were altogether two points of investigation, without intervention, one at the beginning of the school year in October 2008 and one at the end of the school year in July 2009.

Furthermore, there was a questionnaire for teachers and kindergarten educators as well as for parents in order to investigate their beliefs about mathematics and their advancement of mathematical contents either in school or at home.

Tasks

In order to investigate children's knowledge of shapes and to illustrate the concept formation of the children, different tasks were conducted in the interview of which the following will be presented in the paper: (1) naming, explaining and correlating shapes, (2) drawing shapes and (3) identifying and discerning shapes. In the following, the selected tasks presented in this paper are described: For this paper we

(1) Naming, explaining and correlating shapes

At this task, the children were shown different geometric shapes (squares, rectangles, triangles and circles). They were at first asked to name these shapes and then correlate them to a hole in a scarf, which had the shape of one of the geometric figures. Afterwards they were asked to explain a triangle „to somebody who has never seen a triangle before".

(2) Drawing shapes

In order to examine the childrens' transfer from knowledge about a shape into a representation, they were asked to draw a triangle on a paper (c.f. Burger & Shaughnessy, 1986, p. 34f.). Then they were asked to draw another triangle that would be a bit different than the first triangle. After this, again another triangle had to be drawn, differing from the first two. This should be continued as long as it appeared to make sense, meaning so long until the child's way of drawing different triangles revealed something of his or her idea of a triangle and of variety.

(3) Identifying and discerning shapes

Another task giving hints on the conceptualization of the children was a shape-selection task (cf. Burger & Shaughnessy, 1989; Clements et al. 1999, Sarama & Clements, 2009). The children were asked to "put a mark in each of the shapes that is a circle" on a DIN-A3 page of separate geometric figures. After several shapes were marked, the interviewer asked questions such as the following: "Why did you choose this one?", "How did you know that one was a circle?", "Why did you not choose that one?". A similar procedure was conducted for squares, triangles and rectangles and ending with circles and squares in a complex configuration of overlapping forms. The tasks for triangles and rectangles or overlapping forms are not discussed in this paper.

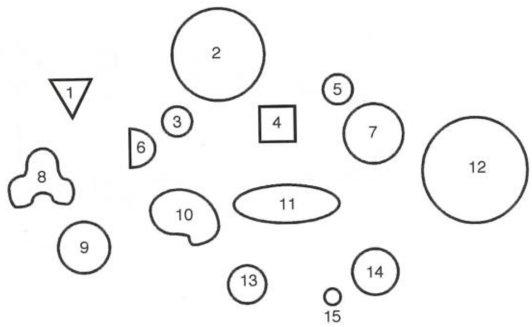

Fig. 1: Student marks circles. From Razel and Eylon, 1991. In Sarama & Clements, 2009, p.269.

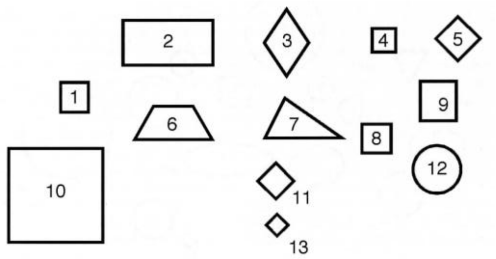

Fig. 2: Student marks squares. From Razel and Eylon, 1991. In Sarama & Clements, 2009, p.270.

Results

In the following, the generation of the categories for the evaluation of the results is presented as well as some results the children achieved at both measuring times, distinct by countries.

Naming shapes

Starting chronologically, it first will be shown what kind of categories could be generated to describe how children named the shapes. Altogether, six categories could be distinguished:

(1) Using terms of comparison instead of the correct name of the shape, for example "like a ball" instead of circle or "like a cupboard" instead of rectangle.

(2) Using terms for solids (3-D-shapes) for 2-D-shapes, e.g. "cube" instead of square or "cone" instead of triangle.

(3) Mixing up terms, using the wrong 2-D-shape-name for another 2-D-shape, for example "square" instead of triangle or "triangle" instead of rectangle.

(4) Using property names instead of the correct shape names, for example "round" instead of square or "acute" instead of triangle.

(5) Using the generic term (quadrangle) instead of the more specific terms (square or rectangle).

(6) Using the geometric terms for each shape.

Table 1: Naming shapes

How do children name the shapes?												
	(1)		(2)		(3)		(4)		(5)		(6)	
	E	D	E	D	E	D	E	D	E	D	E	D
2008	0%	17%	9%	4%	18%	19%	0%	23%	0%	21%	59%	0%
2009	0%	13%	3%	2%	9%	23%	0%	13%	0%	37%	68%	7%

There were several readily discernible trends in the children's developing understanding of shape concepts. The usage of comparative terms only occurred among the German children and the usage of correct geometric terms for each shape occurred more often in England. Only German children used the generic term "Viereck" ("quadrangle"), which is not frequently used in colloquial English. As it becomes obvious in the table above, when children don't know the correct concept, they try to find other logical names for the shapes, as the German children did more frequently.

For the explanations of the children of some shapes, for example a triangle, again several categories could be found:

(A) no explanation given,
(B) gestures used to explain a shape,
(C) comparisons used to explain the shapes,
(D) informal ways of explaining used,
(E) formal ways of explaining used.

Table 2: Explaining shapes

How do children explain the shapes?										
	(A)		(B)		(C)		(D)		(E)	
	E	D	E	D	E	D	E	D	E	D
2008	12%	23%	6%	21%	6%	9%	9%	30%	62%	17%
2009	0%	23%	15%	7%	3%	21%	20%	49%	62%	14%

If we summarize the research findings of this task, it becomes obvious that English children gave more often an explanation or characterization of a shape than the German children. The latter ones used more gestures at the first point of investigation and more comparisons at the second point of investigation, like “this has the shape of a tent or the hat of a witch”. Additionally, there was a bigger tendency in Germany to explain in an informal way, meaning that they tried to explain a shape by its properties but lacked words, such as “side”, “corner” or “straight” or “acute”. The majority of the English children explained the shapes in a formal way, for example “a triangle is a shape with three straight sides and three corners”. However, most English children who didn’t know a formal description did not try to explain the shape in another way.

Drawing shapes

The drawings of the children were thoroughly examined and after several scans and discussions, the following seven categories for the drawings of the children were generated. Here, each child was related to one category.

Category 1: ***Area*** – Child draws triangles in different sizes (but similar angles)
Category 2: ***Angular dimension*** – Child draws triangles with different angles (from very acute ones to obtuse ones)
Category 3: ***Shape*** – Child draws different shapes (correct ones and wrong ones)
Category 4: ***Identity*** – Child draws the same or similar triangle again and again
Category 5: ***Position*** – Child draws triangles in different positions and directions
Category 6: ***Combination*** – Child draws triangles that differ in size, area, angular size and position

Category 7: ***other examples*** – Child draws triangles without one side (just angles) or draws objects from everyday life having geometric shapes (for example road signs).

Table 3: Drawing Triangles

How do children draw different triangles?														
	Cat.1		Cat. 2		Cat. 3		Cat. 4		Cat. 5		Cat. 6		Cat. 6	
	E	D	E	D	E	D	E	D	E	D	E	D	E	D
2008	50%	49%	15%	9%	29%	21%	18%	7%	6%	5%	6%	7%	3%	12%
2009	71%	44%	24%	28%	9%	21%	12%	14%	9%	12%	6%	12%	3%	13%

To summarise the findings of this task, having especially in mind concept formation of the children, it became obvious that most children connected with “different” triangles, triangles that differ in their area dimension but are all pointing upwards and are most of the time equilateral. There were more English children who drew triangles in that category, the English children drew more triangles varying in their shapes (the first triangles were usually correct ones but then other shapes, similar to triangles but for example with wavy sides, were drawn), but later there were more German children drawing triangles in that category. It only occurred in England that the explanation of the triangles did not fit the actual drawing. A triangle was explained, for example, as “having three straight sides”, but in the drawings a shape with three corners and three wavy sides was described as triangle as well, just as a “different” triangle. Triangles as part of the geometric solids in the everyday life (e.g. street signs or tents instead of a simple triangle shape) were only drawn by the German children.

Identifying and discerning shapes

A few of the research findings at the shape-selection task are for example that all the English children could distinguish circles from non-circles correctly. The German children often also marked the oval shape as a circle. At the square-selection task it was the other way round: now, far more German children marked all the correct squares than English children, of which most children only marked horizontal lying squares. Having the children explain their selection, they often argued that “if you turn a square it becomes a diamond”. Selecting triangles, the children of both countries had difficulties, they often didn’t mark triangles that were “upside-down”, but instead triangles with odd sides, with convex or concave sides.

To summarise, concerning the conceptualization of the children, which were examined at the beginning and end of a school year, the following key statements can be made:

(1) German children did improve in their conceptualizations, although they were not formally instructed.
(2) The concepts of the English children were more limited, i.e. more prototype-determined at the end of the school-year than at the beginning of the school year.
(3) The competencies of the children in this research can't be grouped into a hierarchical stage model for the children apply competencies of different stages for different tasks and children of the same age apply competencies of different stages depending on the task.

Discussion

To discuss the research findings, the differences are reflected and it will be hypothesized about possible influences of the different education settings. In England, most of the children gave for example a formal definition of a triangle like "a triangle has three corners and three straight sides". This is exactly what they are taught in school, to give a correct definition for one goal of the foundation stage curriculum is that children should be able to name and explain shapes correctly. However, as was shown before, although the children knew a correct verbal description of a concept, they sometimes had difficulty applying the verbal description correctly. The German children on the other hand did explain the shapes often through comparisons, which they were implicitly shown in the kindergarten. This is also visible in the way they are naming shapes – for they are not taught all the concepts yet, they try to find words they connect with these shapes. Altogether, more English than German children gave an explanation, possibly for they are advanced in school to do so.

Another influence of the concept formation can be the material the children are confronted with. One reason why the English children might only describe a horizontal lying square as a square and one that stands on one of its corners as "a diamond" and not a square anymore, could be the illustrations in the classroom, only showing squares in horizontal position. The reason why they only mark equal-sided triangles as triangles, might be because the material they use for exercising only have equal-sided triangles. But they achieved better results in selecting circles for the prototype perception of one circle is the same as for any circle, for there are only variations in area.

The German children, however, did often not distinguish between a circle and an oval. Therefore, it can be concluded that the input if and how they are instructed as well as the material that is used influences the concept formation of the children.

Conclusion

Still, the question that remains is when would be the best time to actively support the children's geometric concept formation and how should this be done in order to help them to develop a comprehensive knowledge about shapes. This is not easily to be answered. Research indicates that a lot of educational materials introduce children "to triangles, rectangles and squares overwhelmingly in limited, rigid ways" (Sarama & Clements, 2009, p. 216) as was assumed in the research as well, and moreover that "such rigid visual prototypes can rule children's thinking throughout their lives" (Burger & Shaughnessy, 1986; Fuys et al. 1978; Vinner & Heshkowitz, 1980; u.a.). Consequently, teachers as well as kindergarten educators should be aware of the variety of representatives of a certain shape and let them explain what properties a shape needs to have in order to be called "a triangle" for example. An isolated memorising of definitions is seen to be critically and more emphasis should be placed on being able to connect a concept with many representatives as examples. There are findings that one can have a correct verbal description of a concept and possess a specific visual image (or concept image) associated strongly with the concept, but still might have difficulty applying the verbal description correctly (Sarama & Clements, 2009, p. 213). So if we think of in terms of instruction of this mathematical content it should be created in the way that children have the chance to construct a comprehensive concept of shapes .

Literature

Battista, M.T. (2007). The Development of Geometric and Spatial Thinking. In Frank K. Lester (ed.). *Second handbook of research of mathematics teaching and learning: a project of the National Council of Teachers of Mathematics.* 2nd edition. Charlotte, NC: Information Age. R.

Burger, W.F.; Shaughnessy, J.M. (1986). Characterizing the van Hiele Levels of Development in Geometry. *Journal for Research in Mathematics Education.* Vol. 17, No.1, pp. 31 – 48. Published by: National Council of Teachers of Mathematics.

Clements, D.H. (2004). Geometric and Spatial Thinking in Early Childhood Education. In D.H. Clements, J. Sarama (eds.). *Engaging Young Children in Mathematics.* Mahwah: Lawrence Erlbaum Associates.

Clements, D.H.; Battista, M.T. (1992). Geometry and Spatial Reasoning. In D.A. Grouws, D. (ed). "*Handbook of Research on Mathematics Teaching and Learning. A Project of the National Council of Teachers of Mathematics.* New York: Macmillan Publishing Company.

Clements, D.H.; Battista, M.T. (1989). Learning of geometric concepts in a Logo environment. *Journal for Research in Mathematics Education, 20,* pp. 450-467.

Clements, D.H.; Swaminathan, S.; Zeitler Hannibal, M.A.; Sarama, J. (1999). Young Children's Concepts of Shape. *Journal for Research on Mathematics Education*. Vol. 30 (2), pp. 192 - 212.

Darke, I. (1982). A review of research related to the topological primacy thesis. *Educational Studies in Mathematics, 13,* pp. 119. 142.

Dornheim, D. (2008). *Prädikation von Rechenleistung und Rechenschwäche: Der Beitrag von Zahlen-Vorwissen und allgemein-kognitiven Fähigkeiten.* Berlin: Logos Verlag.

Duncker, L. (2010). Methodisch-systematisches Lernen im Kindergarten? Thesen zu einem schwierigen Balanceakt. In G.E. Schäfer, R. Staege, K. Meiners (Eds.). *Kinderwelten – Bildungswelten. Unterwegs zu Frühpädagogik.* Berlin: Cornelsen Verlag.

Franke, M. (2007). *Didaktik der Geometrie in der Grundschule – Mathematik Primar- und Sekundarstufe.* 2. Auflage. München: Spektrum Verlag.

Fuys, D.; Geddes, D.; Tischler, R. (1988). The van Hiele model of thinking in geometryamong adolescents. *Journal for Research in Mathematics Education Education Monograph Series, 3.*

Ginsburg, H.P,.; Opper, S. (1998). *Piagets Theorie der geistigen Entwicklung.* 8. Auflage. Stuttgart: Klett Cotta Verlag.

Gutiérrez, A.; Jaime, A.; Fortuny, J.M. (1991). An alternative paradigm to evaluate the aquisition of the van Hiele levels. *Journal for Research in Mathematics Education,* Vol. 22, No.3, pp. 237 – 251.

Hannibal, M.A.Z.; Clements, D.H. (2008). Young children's developing understanding of basic geometric shapes. *Teaching Children Mathematics*, Vol.5, No.6, FOCUS ISSUE: Geometry and Geometric Thinking (February 1999), pp. 353-357.

Krajewski, K.; Nieding, G.; Schneider, W. (2007). *Mengen, zählen, Zahlen*. Handreichung zur Durchführung der Förderung. Berlin: Cornelsen.

Laurendau, M.; Pinard, A. (1970). *The development of the concept of space in the child.* NewYork, NY: International Universities Press.

Lehrer, R.; Jenkins, M.; Osana, H. (1998). Longitudinal Study of Children's Reasoning About Space and Geometry. In R. Lehrer, D. Chazan, (Eds.). *Designing Learning Environments for Developing Understanding of Geometry and Space.* Mahwah, New Jersey: Lawrence Erlbaum Associates.

Lovell, K. (1959). A follow-up study of some aspects of the work of Piaget and Inhelder on the child's conception of space. *British Journal of Educational Psychology, 29,* pp. 104 – 117.

Page, E.I. (1959). Haptic perception. A consideration of one of the Investigations of Piaget and Inhelder. *Educational Review, 11,* pp. 115 – 124.

Piaget, J. Inhelder, B. u.a. (1975a). *Die Entwicklung des räumlichen Denkens beim Kinde.* Stuttgart: Ernst Klett Verlag.

Piaget, J. Inhelder, B.; Szeminska, A. (1975b). *Die natürliche Geometrie des Kindes.1. Auflage.* Stuttgart: Ernst Klett Verlag

Preiß, G. (2006). *Guten morgen, liebe Zahlen. Eine Einführung in die „Entdeckungen im Zahlenland".* Kirchzarten: Klein Druck.

Preiß, G. (2007). *Leitfaden Zahlenland 1. Verlaufspläne für die Lerneinheit 1 bis 10 der „Entdeckungen im Zahlenland".* Kirchzarten: Klein Druck.

Puhani, P.A., Weber, A.M. (2005). *Does the Early Bird Catch the Worm?* Bericht Nr. 151. Darmstadt: Arbeitspapiere für Volkswirtschaftslehre.

Razel, M.; Eylon, B.-S. (1991). *Developing mathematics readiness in young children with the Agam Program.* Paper presented at the meeting of the Fifteenth Conference of the International Group for the Psychology of Mathematics Education, Genova, Italy.

Rigall, A.; Sharpe, C., (2008). *The Structure of Primary Education: England and other countries* (Primary Review Research Survey 9/1). Cambridge: University of Cambridge Faculty of Education.

Sarama, J.; Clements, D.H. (2009). *Early Childhood Mathematics Education Research. Learning Trajectories for Young Children.* New York: Routledge.

Schäfer, G.E. (2011). *Bildungsprozesse im Kindesalter: Selbstbildung, Erfahrung und Lernen in der frühen Kindheit.* 4. Auflage. Weinheim, München: Juventa-Verlag.

Schäfer, G.E. (2010). (Hrsg.). *Frühkindliche Lernprozesse verstehen: ethnographische und phänomenologische Beiträge zur Bildungsforschung*. Weinheim, München: Juventa-Verlag.

Schneider, W. (2008). *Entwicklung von der Kindheit bis zum Erwachsenenalter: Befunde der Münchner Längsschnittstudie LOGIK*. Weinheim: Beltz.

Szagun, G. (2008). *Sprachentwicklung beim Kind.* Ein Lehrbuch. 2. Auflage. Weinheim und Basel: Beltz Verlag.

Van Hiele, P. und D. (1986). *Structure and Insight: A Theory of Mathematics Education.* New York: Academic Press, Inc.

Vinner, S.; Heshkowitz, R. (1980). Concept images and common cognitive paths in the development of some simple geometric concepts. In R. Karplus (Ed.), *Proceedings of the Fourth International Conference for the Psychology of Mathematics Education* (pp. 177-184). Berkeley, CA: Lawrence Hall of Science, University of California.