Identifying quantities of representations - Children using structures to compose collections from parts or decompose collections into parts

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Abstract

Identifying quantities of collections is a well-accepted content in early childhood education. In this paper different ways of perception and determination of quantities of collections will be discussed because they shall contribute in different ways to the development of conceptual and procedural understanding of numbers. First, we will look on the different processes which are used for identifying quantities theoretically. The processes of perception, in particular the process of decomposing a collection into parts, will be focused on. Next, it will be investigated how children recognize or perceive collections of objects. We will ask whether and how they decompose or structure collections of single objects into substructures or parts. On the basis of the results of the present study it will be concluded how this ability can be supported and can constitute a basis for formal mathematics in school.

Introduction

Identifying quantities of collections is a well-accepted mathematical content in early childhood education. Many pre-school teachers think immediately of the process of counting every single item as a way to determine the quantity of the representation of the collection (Benz, 2010). Counting is important in early mathematics education and an obvious learning goal for all professionals in early childhood and can be seen as a milestone in the learning process However, counting is not the only way to determine quantities, as we see below. If we analyse the different processes which can be used for identifying quantities of representations it becomes clear that there are other valuable competences which can also be focused on.

Theoretical Background – Different processes to identify the quantities of collections

In order to describe different aspects and competences in identifying quantities of representations of concrete objects here we will theoretically distinguish between two different steps (Benz 2011; Steffe & Cobb, 1988):

Step 1: The process of *perception* of the representation of the quantity.

Step 2: The process of *judgment* or *determination* of the whole quantity of the collection.

The first step – *the process of perception* – will be discerned in three different kinds of perception and then we will assign the various processes of determination.

Table 1: Different processes in identifying the quantity of a collection

Step 1: *Process of perception*	*Step 2:* *Process of determination*
Quantity as a collection of single objects only	Counting every single object - *Counting all*
	Subitizing for sets with quantities up to 3
Quantity as a whole	*Subitizing* • Because the figure or pattern of the representation is already known (e.g. dice patterns up to 6 or finger patterns)
Quantity as a composition of different parts • Identifying the parts • Structuring the quantity in different parts or substructures	*Counting* every single item of every part – no subitizing of the parts
	Conceptual subitizing *Subitizing* one part or all parts • *Counting* every item of the whole quantity • *Counting* only the second part and starting with the quantity of the first • *Counting in steps*
	Conceptual subitizing *Subitizing* all parts and knowing the results

As one possibility of perception the collection can be seen as a conglomerate of lots of single objects. Then there are different possibilities to determine the quantity. Every single object can be counted or if the collection has a small quantity it can be determined with subitizing. *Subitizing, spontaneous subitizing, perceptual subitizing* or *simultaneous recognizing* means "recognizing a number without consciously using other mental or mathematical processes and then naming it" (Clements & Sarama, 2009, p.44). Still there are different theories about the mental processes which are behind the ability to subitize but "regardless of the precise mental processes subitizing appears to be phenomenologically distinct from counting and other means of quantification" (Clements & Sarama, 2009, p.44). Even if research results vary how many objects can be subitized at once no one speaks about subitizing a set of more than six objects (Clements & Sarama, 2009). So the process of subitizing is limited to a small number of objects. Some research suggests that sets with more than 3 objects will be decomposed and recomposed without the person being aware of the process (Clements & Sarama, 2009, p.45).
If the collection is perceived as a whole figure and the figure will be recognized immediately because it is well known or memorized like dice or finger patterns some researchers speak also of subitizing. But it is not sure if the children are indeed aware of the quantity of the single items of this arrangement and that this representation is a composition of different parts (like 4 and 1). They could also just have learned the "name" of the figure without being aware of the quantity (von Glasersfeld, 1987, p.261). Next, to perceive the quantity as a whole entity there is another way of perception of a representation of collections. A set of collections can be decomposed through structuring this collection and identifying different parts in this col-

lection. The idea of structuring and decomposing a representation in different parts can lead to different ways to determine the quantity of a collection.

After identifying different substructures or parts still every item can be counted. Another possibility is to perceive one or two parts of the substructures with subitizing and then counting every item or counting only the second part and starting with the quantity of the first part or knowing the result.

The mental act of decomposing a collection in its constituent parts can also be described as identifying, seeing, perceiving or creating a structure in the collection so that different parts or substructures can be identified. Sometimes the arrangement of the objects or the spatial structure of the collection can lead to the grouping but still the identification of the structure is an individual act (Söbbeke, 2005). Structuring a quantity into different parts or substructures is seen as a powerful mathematical activity. In previous research and mathematical theories different reasons for supporting the perception of structures and the ability of decomposing a collection into parts are evident and will be discussed in the next paragraph.

The importance of perception of structures – decomposing a collection into parts

In terms of a part-whole understanding decomposing a quantity into parts or substructures is an important ability. Resnick (1983) points out that an interpretation of numbers in terms of part and whole relationships is very important. She mentions that a primitive form of part-whole reasoning occurs in early counting routines when children are able to maintain a partition on a collection of items: those items already counted and those items yet to be counted. She proposes that later on the basis of this basic part-whole schema a *quantitative* part-whole schema will be established which can be observed when dividing a collection into parts. Gaidoschik (2010) and Young-Loveridge (2002) point out the connection between the structuring of a collection of items and the development of a quantitative part-whole understanding. The quantitative part-whole understanding is to be seen as one important component for building mental calculation strategies, another important step in school (Gaidoschik, 2010; Gerster & Schulz, 2004).

The competences in part-whole-understanding and the competence of perception of structured quantities are – next to advanced counting competences – evaluated as predictors for arithmetical competences in year 2 (Dornheim, 2008). The particular relevance of identifying structures in collections of quantities was also investigated by Mulligan et al. in the project Awareness of Mathematical Pattern und Structure AMPS (Mulligan et al., 2010). They were able to show that children who are low achievers in mathematics had problems to perceive structures in visual representations (Mulligan, 2002). A general connection between awareness of structures and pattern and mathematical abilities is stated by Mulligan and Mitchelmore (2009, p.35).

A link between spatial structuring abilities of children at age 4-6 and developing number sense is suggested by van Nes (2009). She investigated spatial structuring ability in different tasks with 38 children at the age of 4 to 6 years and postulated 4 phases in spatial structuring ability. In these phases she also focused on the ability to produce structures in unordered quantities with reference to determination of quantities. Lueken (2010) interviewed 74 first graders (age of 5;8 to 7;2) at the beginning of school with a semi-structured interview about early structure sense. The interview contained tasks in visual, tactile and audio patterns and asked for explanations and reproduction of structured didactical material which is used in primary school like a *ten-chain* and the *twenty-field*. The results of Lueken's study showed that there is a correlation between early structure sense and mathematical competences, tested with the standardized OTZ test 3 months before the children entered school. She also showed that an early structure sense can be seen as a predictor for mathematical achievement at the end of year 2.

These theoretical and empirical studies show evidence for the relevance of identifying structures in representations and the connection with children's arithmetical development. Many researchers of these studies indicate that their research is done with a small group of children in special settings, thus the analyses are rather exploratory than confirmatory and offer only trends.

In most of the reported studies the abilities to structure small quantities were tested mainly by reproducing structured representation or by determining their quantity when the representations were shown only for a short time to the children. This made it necessary to use subitizing for determining the quantity of parts or the whole. The memory can also play an additional role because the different parts must be memorized for determination. In this paper, considering the difference between the processes of perception and the determination of quantities, we focus on the process of *perception* of quantities. It will be investigated if and how children at age of 4-6 perceive (de)composition or structures of quantities in representations as a help for determination. This leads to the following research questions:

1. Do pre-school children perceive arranged structures in collections and can they use this perception to determine the quantity?

2. Do pre-school children use the idea of decomposing or structuring a collection into parts to create a representation of a collection so that other people can easily see how many objects are there? What kind of structures do they use?

Most of structured didactical material which is used later on in school is designed with structures of parts of 5 and 10 in order to enable children to (de)compose quantities easily for perception. This kind of structure is an important structure in terms of perception for numbers up to 100. Because perception of structures even in structured representations is an individual act, a third research question will be raised with reference to the idea on building further competences on informal strategies:

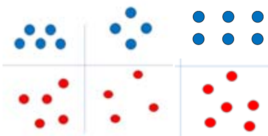
3. What kind of structures do children use or perceive in a ten-frame to represent a collection so that other people can easily see how many objects are presented?

Design

Although it was planned to interview only children at age 4 to 6 some younger children asked to be interviewed as well so they were included in the study. Altogether 189 children at the age of 3 to 6, all attending German kindergarten, were interviewed individually. In the interview they had to solve different tasks. The interviews were conducted in two parts to avoid too much strain on the children. The children could stop the interview at any time, thus not every task was solved by every child. All tasks were posed in the same order. The interviewers took care to ask the children also whether they could explain why they solved the task in the way they did. The solving process was videotaped and later on transcribed. Later on, a qualitative and quantitative analysis of the solutions and explanations was undertaken. To answer the research questions above the analyses of three tasks of the interview were selected.

To investigate the first research question – do children at the age of 3 to 6 perceive arranged structures in representations of quantities and can they use this perception to determine the quantity – a task of the study of Gasteiger (2010) was chosen. Here, the children were given cards with blue and red dots.

Diagram 1: Task "Grouped Quantities"



Then they were asked to find a card with blue dots that corresponds to a card with red dots: *For every blue card there is a red card. Do you have an idea, which of these cards belong together?* This question does not address the perception of structures directly. With this question it is investigated if children focus on the aspect of quantity in general. The children had to identify the quantity so that they could reflect about the process of perception. If the children did focus on the aspect of quantity and tried to make pairs with the same quantity they were asked: *On which cards could you identify easier how many dots there are*? Then they were asked to explain their opinion. The children could take as much time as they needed.

With the second task the children's ability to decompose a quantity into parts was investigated in a reversible way. Therefore, it was examined if children at the age of 3-6 already use the idea of decomposing or structuring a collection into parts to create a representation of a collection so that other people can easily see how many objects there are. It also was investigated what kind of structures the children used. First the children were asked to create a collection with 7 counters so that another person easily can see how many items there are. Then they were asked why they think it can be easily seen how many counters are on the table.

The collections were categorized in different categories. If the children put the counters in a row so that no decomposition was clearly shown it was categorised as *not structured into parts*, also if they created a circle. If any decomposition into parts could be seen so that someone had the chance to determine the quantity without counting every single item, it was categorised as *structured representation.*

In order to see what kind of structures young children see and use when they deal with structured material of formal school mathematics the children were also asked to sort 5 eggs in an egg carton. In Germany egg cartons usually contain 6 or 10 eggs. We used a carton for ten eggs because it is equivalent to the didactical material of a ten-frame (Gerster, 2004). Here, the children are forced to use the structure of the ten-frame. Still, it was in our interest to investigate how they "use" the structure to create quantities which can be seen easily and whether they can explain afterwards if and how they used the structure of the ten-frame for (de)composition. They were asked to put 5 eggs in a carton so that it can be easily seen and then to explain their representation. In the analysis the structures are described. To categorize the explanations it was investigated if and what kind of structure or (de)composition the children referred to.

Results

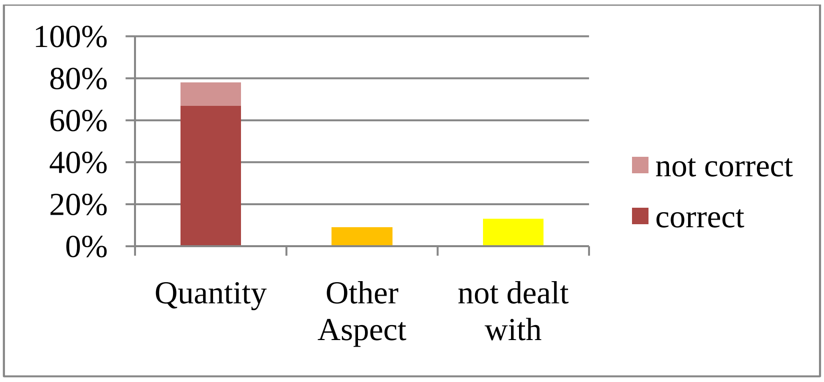
Perception of structures as a help for determination quantities

67% of the children made pairs with the same quantity correctly.

11% of the children tried to make pairs with the same quantity but not every pair was correct.

9% of the children made pairs on the basis of other criteria (like "nice – not nice"). 13% of children did not deal with the task; (e.g. they had no idea or said, "I cannot do that").

Diagram 1: Aspects of correspondence



It can be stated that many children (78%) between the age of 3 and 6 see quantity as a criteria for correspondence and that they can make right correspondences in terms of same quantity.

In the following table it is illustrated how it looks like if the children are divided into age groups:

Table 2: Aspects of correspondence – Children at different age groups

Age of children (year; month)	N	Pairs of same quantity		Other correspondence	Not dealt with
		Correct	Not Correct		
3;6-3;11	8	3 (37,5%)	-	3 (37,5%)	2 (25%)
4;0-4;11	74	44 (60%)	10 (13%)	6 (8 %)	14 (19%)
5;0-5;11	87	63 (72%)	10 (12%)	5 (6%)	9 (10%)
6;0-6;11	20	16 (80 %)	1 (5%)	1 (5%)	2 (10%)

With increasing age the children perceive quantities rather as criteria for correspondence. Also with increasing age the number of correct pairs of the same quantity rises.
If the children did focus on the aspect of quantity and tried to make pairs with the same quantity they were asked: *On which cards was it easier to identify how many dots there are?*
If we look only at the children who dealt with that task in terms of focusing on the same quantity it can be stated that 75% of the children who focused on the same quantity responded that they could identify the quantity easier at the grouped representations. 16% of the children who focused on the same quantity preferred representations without arranged structures and 9% of the children who focused on the same quantity did not see any difference.

Then the children were asked to explain their decision:
27% of the children who focused on the same quantity gave explanations which didn't refer to the arrangement or structure like *I can already paint dots*. Some of them made a connection to colours like *Blue is my favourite colour* or *I wear a blue T-shirt.*
54% of the children who focused on the same quantity did refer in their explanations to the structure in the representations. Different aspects of the arrangements are described:

- Structure in general
  The red cards are disordered
  The blue ones were easier painted
  The blue cards are more correct
- Dice pattern
  The 4 and the 6 look like a correct 4 and 6
- Describing the structure – Quantity of the parts
  (Card with 4 dots): *2 and 2*
  (Card with 5 dots): *If you look skewed, you can see 2 and 2 and then 1*
  (Card with 5 dots): *Because on the bottom there are 3 and above there a 2*
  (Card with 6 dots): *Here are 3 and here are 3 too*

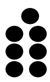
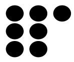
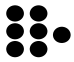
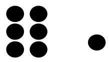
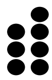
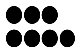
The results show that pre-school children can already perceive structures in representations and they can use them to determine quantities. However, for interpreting these results in terms of ability of decomposing a collection into parts through structuring, it must be considered that the cards with the 4 and the 6 dots were very similar to dice patterns. It cannot be stated clearly if they perceive these collections of quantities as a structured composition of different parts or as a whole figure which they recognize again as memorised pictures or figures (von Glasersfeld, 1987). Thus, for the next task the quantity of 7 counters was chosen so that not only one dice pattern as a whole figure could be reproduced.

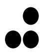
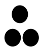
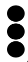
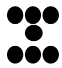
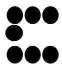
Structures used to create representations
The children were asked to create a representation with 7 counters so that other people can easily see how many objects there are. The counters had all the same colour so that a composition only can be demonstrated through spatial structures. The children were not given 7 counters; they first had to count 7 counters out of a bowl with many counters.

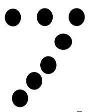
Table 3: Representations which can be seen easily

		Structured representation	Not structured into parts	Wrong quantity represented	Not dealt with
All Children	189	108 (57%)	43 (23%)	22 (12%)	16 (8%)
Age of children (year; month)					
3;6-3;11	8	3 (37,5%)	4 (50%)	-	1 (12,5%)
4;0-4;11	74	34 (46%)	14 (19%)	14 (19 %)	12 (16%)
5;0-5;11	87	56 (64%)	21 (24%)	7 (8%)	3 (4%)
6;0-6;11	20	15 (75 %)	4 (20%)	1 (5%)	-

Here the same tendency can be seen as with the first question. Many children, especially the 5 and 6 years old children, already used a structure to create a representation which can easily be seen. They decomposed the collections in different parts.
Most children used a composition of the dice pattern of 6 and then placed next to the dice pattern one counter. Here I do not distinguish the different possible orientations.
(Number of children who created this representation is in brackets)

(23)  (11)  (5)  (5)  (20)  or 

Most of the children (51) explained their (de)composing with explanations like *One more than six.* For the last representation in this line the children gave different explanations because they used different structures. 7 of the 20 children explained their representation as a composition of 6 and 1 and 13 children saw a substructure of 3 and 4. The structure of 3 and 4 could be seen with other representations too, whereas the 3 and the 4 were represented differently. They sometimes used the dice pattern for 4 but no child used the dice pattern for 3: (3)   (4)   (4)   . Altogether 24 children used the composition of 3 and 4. The composition of dice patterns of 5 and 2 was created by 12 children. Other representations which were used quite frequently divided the dice pattern of six in to rows and then one point was put between the rows: (6)  (4) 

9 children placed the counters so that the digit was represented:  Here the children did not use the idea of structuring a quantity but rather the idea of using digits to describe quantities which is an obvious solution for the task.
In this paper not all compositions are reported but only the most frequent compositions.
Summing up, it can be stated that children at pre-school age are able to decompose representations of quantities in different parts to facilitate for other people the perception of the quantity whereupon in most of the structured representation dice patterns were used in some way.

Different perceptions of structures in structured material

In order to see what kind of structures young children perceive and use when they have to deal with structured material with a structure of 2x5, the children were asked to sort 5 eggs in an egg carton which looks like a ten frame. Thereafter they were asked to explain why it can be seen easily. Through the structure of the ten-frame they could not reproduce the dice pattern of 5 which some already know as a whole figure. So they were forced to find another way of representation.

Table 4: Structures of 5 eggs in an egg carton

		5 in a row	●●● ●●	●●●● ●	Other structures	Wrong quantity	Not dealt with
All Children	189	77 (41%)	66 (35%)	12 (6%)	10 (5%)	8 (4%)	16 (9%)
Age (year; month)							
3;6-3;11	8	1 (12,5%)	1 (12,5%)	-	-	-	6 (75%)
4;0-4;11	74	35 (47%)	24 (32%)	1 (1%)	2 (3%)	5 (7%)	7(10%)
5;0-5;11	87	37 (43%)	33 (38%)	7(8%)	4 (5%)	3 (3%)	3 (3%)
6;0-6;11	20	4 (15%)	8 (45%)	4 (20%)	4 (20%)	-	-

Most children put the 5 eggs in one row; one third of the children put three eggs in one row and two eggs in the other row. These were the most frequent representations.

With increasing age the use of this structure ●●● ●● is more frequent and the representation with 5 eggs in a row decreases. How the children perceive their representation cannot be concluded from their representation because the perception of a structure and (de)composing the quantity into substructures and different parts is an individual constructive act. The given structure of the ten-frame automatically produces a kind of structured representation. But if the children perceive the quantity as a conglomerate of single items, as a whole figure or as (de)composition of different parts with a structure cannot be answered only through interpreting the created representation. Therefore, the interviewers asked the question *Why do you think, it can be easily seen, that there are 5 eggs?* in order to get an indication what kind of structures the children perceive in their representation:

Table 5: Answers referring to structures or compositions

Answers referring to structures (Explanations in grey rows refer to a (de)composition)	*5 eggs in row*	●●● ●●	●●●● ●	*Other structure*
No explanation	N = 36	N = 12	N = 1	N = 0
Referring to no structure – Mention of counting every single egg	N = 28	N = 4	N = 1	N = 0
Referring to the row without reference to the ten-frame	N = 5			
Structure of a ten-frame	N = 5			
(De)composition in 2 and 3	N = 3	N = 8	N = 3	N = 3
(De)composition in 2, 2 and 1		N = 4	N = 3	N = 2
(De)composition in 4 and 1 without explicit mention dice pattern		N = 6	N = 4	N = 3
Dice pattern of 4		N = 13		N = 1
Dice pattern of 5		N = 12		N = 1
Dice pattern of 6		N = 7		
	N = 77	N = 66	N =12	N = 10

As it can be seen in table 5 most of the children using the representation of 5 eggs in a row did not refer to a composition of a quantity in different parts. 64 children gave no explanation or gave an explanation which can be interpreted as a perception of the quantity as single eggs because they counted every single egg. But we cannot be sure if this really is the case. It is just an assumption. Only 10 children referred to the structure as a row. 5 of these children used the structure of the ten-frame in their explanation like *It is a carton for 10 eggs and therefore 5 eggs are in a row* or *10 eggs are in the carton and 5 is the half.* At this point it must be stated that the structure of the ten-frame like it is often used in primary school, where the quantities are perceived as composition of two quantities of 5 items, is not used very much by children at the age from 3-6. 3 children explained that they perceive a (de)composition of 2 and 3 in a row.

50 children of the 66 children using the representation ●●● ●● referred in their explanations to a structure. Only 16 gave no explanation or referred to counting every single object.

8 children perceived the two rows as a division into the parts in 2 and 3. The structure of the dice patterns of 4, 5 or 6 were mentioned in most of the explanations referring to a structure.

Children using the representation ●●●● ● also referred proportionally quite frequent to a structure in their explanation.

Discussion

In the theoretical background different processes in identifying quantities are discerned. In this paper I tried to investigate the process of perception, especially the perception of structures in order to identify different parts in a collection. It is not easy to gain insight how children perceive a collection of objects – because there is no obvious action to observe. Therefore, we only can draw conclusions out of the explanations of the children or through careful interpretation of their way of determination or (re)producing representations, still having in mind that there is no one-to-one correspondence between perception and determination. In summary it can be stated that half of all children of this study at the age of 3-6 already discern between structured and not structured representations and that they can use this perception to determine quantities. As already mentioned at the first task not only the perception of structures but also the perception of quantities as a whole figure was investigated because the representation of the quantity 4 and 6 was similar to dice patterns which are mainly recognized first as whole figures. Some children already referred in their explanation to the quantity as a decomposition of parts even for the dice-like representation of 4 and 6. In the reversible task where children should create a representation of 7 items the children had to (de)compose the quantity because a dice pattern of the quantity of 7 does not exist. Here the same tendency can be seen: More than half of the children already used a structured composition for their representation. With increasing age the perception and the use of structure increased. Looking on the structures and decompositions which are used or explained by the children for the quantity of 7 the decomposition of the dice pattern of 6 was dominant. A possible interpretation for the preference of this use can be the fact that children used 6 as the next number to 7. Another possible interpretation can be that 6 is the largest quantity which can be represented with one dice pattern and the children used the largest possible quantity. Interestingly, the composition of 5 and 2 was used less frequently as the composition of 3 and 4. A possible explanation for the preference of the composition of 3 and 4 can be the proximity to 3 and 3. The (de)composition 3 and 4 then can be seen as nearly dividing into halves or nearly doubling (Rottmann, 2006). The structure of the finger-pattern with 5 and 2 was not transferred to create a structured representation with 7 items. In this setting it has now to be noticed that the quantity of 5 did not play a big role in children's ideas of (de)composition with round counters. Looking at the explanations of the children about the structures they used for the representation in the ten-frame it can be stated that most of the children who used the structure of 5 items in a row in the structured ten-frame as a structure did not explain the representation of the quantity with reference to a structure or (de)composition. If they gave an explanation they referred to counting every single item. This can be interpreted in the way that they did not use a structure or (de)composition to perceive the quantity with different parts. Children who explained the quantity with (de)composition either used the two rows to structure the quantity in two parts or they referred to dice patterns. Most of the children didn't perceive and use the structure of a ten-frame in the conventional way of perception which refers to decomposition the ten in two parts of 5 and refers to a row as a whole quantity of 5 items. The composition in parts of 5 and 10 is used in most didactical materials to represent numbers up to 100, so that conceptual subitizing is possible. If the children in this study used the row for the 5 eggs most of the children had to count. Only very few children could "use" the knowledge of the fact that there are parts in terms of rows which are representing 5 items in this ten-frame. This is not astonishing. It emphasizes the fact that children's perception of structures is an individual act. Also it becomes clear that the decomposition and perception of quantities in parts of tens, fives and ones which is used in didactical material has to be learned.

Conclusion

On the basis of the children's created representations of quantities and their explanations of their way of determination the quantity of the representations it was carefully tried to interpret how the children probably could have perceived the representation, still having in mind that there is no one-to-one correspondence between perception and determination. Many children could explain structures in quantities and they also could explain how they used the idea of (de)composing a representation of a quantity into different parts for determination. Therefore; it can be concluded that children at the age of 3-6 already can perceive structures in representations of quantities and use this for to determine the quantity. Focusing on the perception of structures seems to be an adequate mathematical content for children in pre-school education. However, it must be noted that there are also many children who do not obviously perceive or use structures. Therefore, it is a challenge for professionals to support children in the perception and usage of structures through questions and reflections about the perception of structures and through providing materials where children can perceive structures (like egg cartons). Regarding the idea of building formal knowledge on informal strategies professionals must have in mind that many children perceive their own structures in didactical material, so the different perception has to be a point of discussion even when using didactical material with an arranged structure.

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