The use of interactive visualizations to foster the understanding of concepts of calculus – design principles and empirical results

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Abstract

Calculus was introduced to the German curricula as a result of the so-called 'Meraner Reform' of 1905. The 'education to functional thinking' was formulated as a specific task within this reform. Functional thinking was meant in a broader sense: as thinking in variations and functional dependencies spanning the whole mathematics education in terms of a fundamental idea. Especially the dynamic aspect of functional thinking was pointed out. Calculus was seen as culmination point reached by an organic structured mathematics education, and functional thinking can be seen as propaedeutics to calculus (Krüger 2000). Calculus at school is often procedure-oriented, and structural understanding is lacking. The presented work is meant as a contribution to a qualitative structural-oriented approach to school calculus. In this context several interactive learning activities based on Java applications were developed. By interactively visualizing functional dependencies simultanously in different representations the students are enabled to explore various aspects of functional dependencies. The activities emphasize both the dynamic aspect as well as the object view of a functional dependency by using a double-stage vizualisation.

The activities were used in a qualitative study with 10th grade students (age: 15 to 16) in different secondary school classes in Berlin, Germany. Based on the analysis of video observations some results of the study are presented.

Keywords

interactive learning activity, functional thinking, calculus, qualitative study

1 Introduction and Motivation

To introduce to the topic and to point out the objective of the presented work an example will be presented. The following problem was given to more than 100 students.

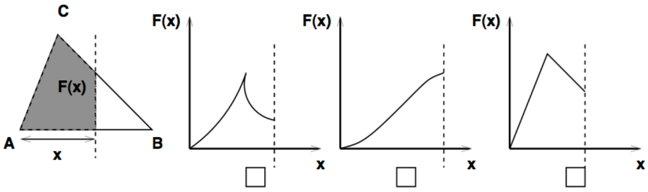

Figure 1. The dashed line moves rightwards. F(x) is the area of the grey part of the triangle dependent on the distance x. Which graph fits and why? (Schlöglhofer 2000)

The main mistake was to put a cross on the graph on the right side together with a reason like "The area of the graph is just like the area F(x)".

This *graph-as-image-misconception*, where the graph is interpreted as a photographical image of the situation, results from an unsufficient dynamic view of the functional dependency. Usually in class the pointwise view of functions is predominant, whereas the dynamic view is underrepresented.

The student solution below shows why this example leads to fundamental concepts of calculus.

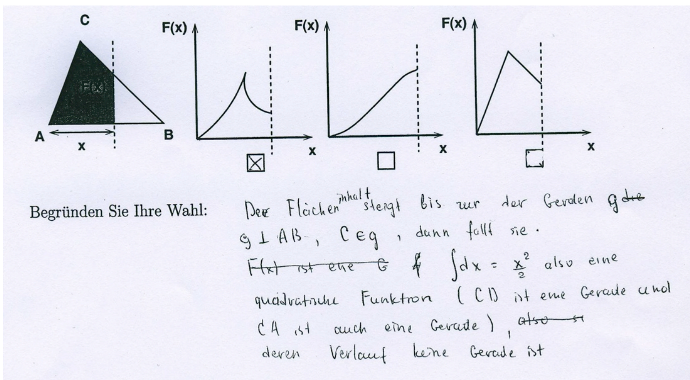

Figure 2. Student reasoning (translation): "The area increases until the line g, g $\perp$AB, C$\in$g, then it decreases. $\int dx = \frac{x^2}{2}$ thus we have a quadratic function (CD is a straight line and CA ist a straight line also), whose run is not a straight line".

This student used the concept of integration by identifying the triangle with a piecewise linear function whose antiderivative must be quadratic. This answer is still combined with the graph-as-image-misconception, because the chosen graph decreases after C is passed. The student uses the concept of integration without activating a dynamic view of the function.

The mathematical background of this example is the *Fundamental Theorem of Calculus*: The triangle interpreted as piecewise linear function is the derivative of the area function F, and the area function is an antiderivative. Besides monotony of the area function there is an inflection point where the quality of the growth changes: the growth of the area increases until this point and decreases afterwards. Note that the inflection point in this example cannot be found by applying the common routine using the second derivative, because the piecewise linear function is not differentiable in C.

1.1 Functional thinking and propaedeutics to calculus

Calculus was introduced to the German curricula as a result of the so-called *Meraner Reform* of 1905. The *education to functional thinking* was a main task of this reform. Functional thinking was meant in a very boad sense: as thinking in variations and functional dependencies – always with a view to the aspect of change. It should span the whole mathematics education in sense of a fundamental idea. Calculus was seen as a culmination point in mathematics education and the education to functional thinking as propaedeutics to calculus (Krüger 2000).

Vollrath (1989) describes three aspects of functional dependencies: static aspect (functions as pointwise relations), aspect of change (dynamic view of a function), and object view (functions as objects, as a whole). The last two aspects are close to the meaning of functional thinking of the Meraner reform, but do not cover the term sufficiently.

The underlying term of functional thinking for the presented work is based on the term of the Meraner reform: functional thinking as propaedeutics to calculus with emphasis of the aspect of change and variation.

1.2 Problem situation at school

In the following paragraph some typical problems of students connected to functional thinking and calculus at school are described.

Problems with the definition

Usually at german schools the students learn the following definition for a function (based on Dirichlet):

A function is an assignment rule by which each element x of a set is assigned to exactly one element y of (another) set.

This definition usually appears in grade 7 (student age: 12 years) and is introduced by investigating examples and non-examples. But the above definition is abstract and very general. There are many aspects and associations that are not 'covered' by this definition. The value of functions becomes clear not until questions of calculus are addressed. Therefore the Dirichlet definition is not really understandable in grade 7. Since the students only get to know a few classes of functions (starting with linear functions, quadratic functions etc.) the generality of the definition does not seem to be appropriate (Fischer and Malle 1985). The 'limited' experiences of the students form the *concept image* of functions. That is the set of all mental images and characteristics connected with the concept of function. Whereas the *concept definition* is the definition for function one would give (Vinner and Dreyfus 1989). Vinner and Dreyfus (1989) pointed out that students make decisions about whether a given example is a function or not based on their concept image and not on the concept definition. The concept image often contains images like: a function has to be *one* rule, has to 'look reasonable', is injective, does not have any jump discontinuities. The concept image and concept definition are often non-consistent. This is related to the work of Sierpinska (1992) who describes several epistemological obstacles in the learning process in the field of functions like: the illusion of linearity, physical laws have nothing to do with functions, functions are tables and therefore not worth to be studied as objects, etc.

Calculus at school

Calculus at school is still mainly procedure-oriented and consists in curve sketching routines rather than structural understanding. Many authors describe this problem (Blum and Kirsch 1979; Bender 1991; Hahn and Prediger 2008, etc.) and claim a qualitative-structural approach to school calculus - a claim which is in fact 100 years old and was already formulated in connection with the Meraner reform (Krüger 2000; Weinmeister 1907).

1.3 Objectives

The presented work is meant as a contribution to a qualitative-structural approach to school calculus using the computer. Based on the IGS Cinderella (http://www.cinderella.de) interactive learning activities were built, and design principles emphasizing the dynamic and object view of functions where developed. The activities were used in a qualitative study in german secondary schools in grade 10 (age 15/16), before the students learnt about curve sketching routines. Two of the activities, the design principles and some results of the study are presented below.

2 Interactive activities – basic ideas and design principles

Basic idea is the use of an interactive visualization which accentuates the dynamic component of functional thinking and aims at the development of qualitative-structural concepts with regard to propaedeutics of calculus.

In this context three learning activities were developed in joint work with Andreas Fest, PH Schwäbisch Gmünd. The activities consist of Java applets embedded into a webpage and can be used with a standard internet browser (*low technical overhead*). The activities and related teaching material (worksheets etc.) are accessible on http://www.math.tu-berlin.de/~hoffkamp.

2.1 Design principles

The design principles are also described in Hoffkamp 2009 and 2010, but an overview will be given in the following. Figure 3 shows the learning activity 'Dreiecksfläche' ('area of a triangle').

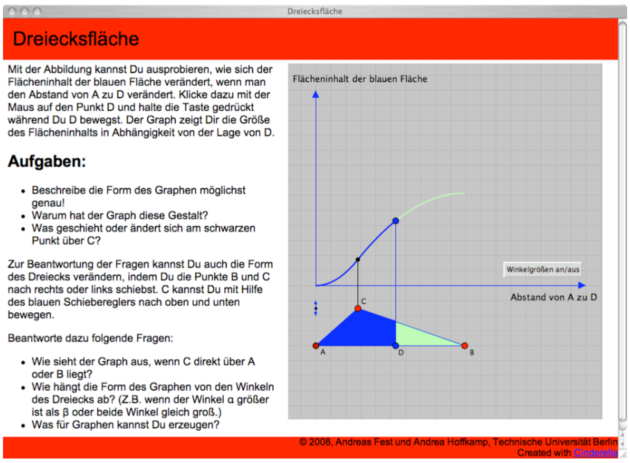

Figure 3. Screenshot of the interactive learning activity 'Dreiecksfläche' ('area of a triangle').

The activity allows the students to explore the functional dependency between the distance AD and the dark area within the triangle. Moving D visualizes the dynamic aspect of the functional dependency simultaneously in situation and graph. Moving B and C changes the triangle and the graph as a whole. For example the property monotony is now very easy to formulate: 'There is more and more dark area added when moving D to the right.' Especially the inflection point is dynamically observable as a point where the quality of growth changes and students find formulations like: 'After the black point over C the added area decreases'. Note that the area function F is piecewise quadratic, and does not belong to the common classes of functions taught at school.

Starting point of the activities is always a dynamic *connection situation-graph*. The representation form *graph* was chosen because it relates in an eminent way to the dynamic aspect of functional thinking and contains the information about the function 'at a glance'. Each activity contains *two levels of variation*: Moving D allows *variation within the situation*. The aspect of change becomes simultaneously visual in both representation forms - situation and graph.

The second variation level, which will be named *metavariation* in the following, allows - by moving points B and C - to change the situation itself and the function as a whole. Therefore it relates to the object view of the function. Metavariation forces the detachment from concrete values and leads to a qualitative view of the functional dependency and its local and global characteristics. Characteristics like 'monotony' or 'existence of an inflection point' are emphasized by being invariant (or in case of the inflection point 'nearly invariant') under metavariation.

The role of *language as a mediator* between the representation and the mental images of the student is important when working with the applets. The students are always asked to verbalize their observations.

The *principle of contiguity* (Mayer 2005) is essential. Representations referring mutually are close to each other in space and time. Especially the fact, that movement takes place where one operates with the mouse is important to achieve an integrative visualization.

2.2 The role of visualizations in mathematics and mathematics education

To some extent mathematics takes place in an interaction between representation, interpretation and operation (Fischer and Malle 1985). The operational side is still predominant in mathematics education. But operation requires representation and description, because one operates with symbols that are abstract descriptions of interrelations. Mathematicians do not carry out abstractions in the mind. They search for a physical representation of abstractions. This is an essential property of mathematics. Visualizations and physical representations allow extended elaboration by focusing on certain aspects – in case of the activities by focusing on the dynamic view within the transfer of two functional representations. This is also important for the ability to communicate visually. Moreover, while language and writing is linear and complex issues need to be brought in a succession, visualizations like the presented activities allow an integrated image by presenting the issues side by side as a whole.

Learning activity 'Einbeschriebene Rechtecke' ('inscribed rectangles')

Figure 4 shows parts of the activity 'Einbeschriebene Rechtecke' ('inscribed rectangles').

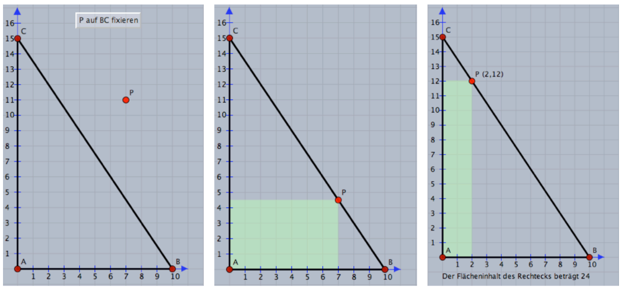

Figure 4. Applet constellations within the first part of the activity 'Einbeschriebene Rechtecke' ('inscribed rectangles').

Students are asked to move point P on the line BC so that an inscribed rectangle appears. After fixing P on BC by using a button, P can be moved on BC and the area of the rectangle dependent on the x-coordinate of P can be explored by answering questions like: 'Describe how the area of the rectangle changes, when moving P on BC.', 'Which values does the area take? Also 100?', 'Are there different rectangles with the same area? If yes, give examples for it.' When using a 'hint'-button the coordinates of P and the value of the area appear.

Figure 5 shows the applets within part two and three of the activity.

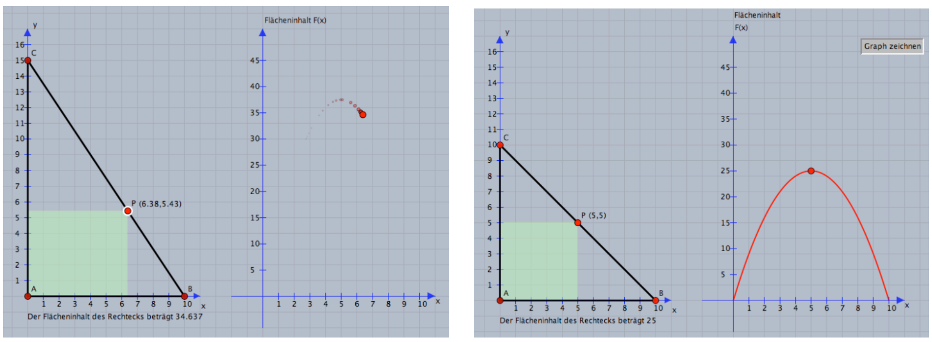

Figure 5. The two levels of variation within the activity 'Einbeschriebene Rechtecke' ('inscribed rectangles').

The left screenshot in figure 4 shows the variation within the situation. Moving P on BC leads to simultaneous movement of the point in the area graph leaving a trace. Students are asked to answer the questions: 'When is the area at its maximum?', 'Why is the area graph shaped like this?', 'Explain which values for the area do not appear, appear once, twice using the graph.' On the right side the applet referring to metavariation is shown.

Students are asked to move B and C on the axes and describe the influence on the graph. Two main questions are: 'Give two examples for triangles so that the rectangle with maximum area is 15. Describe how you found these triangles.' and 'Give two examples of triangles so that the maximum is taken at x=4. Describe how you found these triangles.' Note that the situation in figure 5 shows an isosceles triangle in which case the rectangle with maximum area is a square.

3 Qualitative Study

3.1 Research questions

The following research questions conducted the study:

1. Which terms and mental concepts with regard to a dynamic view of functional dependencies do the students develop when working with the activities, especially when local and global characteristics of the functions are described qualitatively in connection to concepts of calculus?
2. What forms of interaction processes can be found and what is the role of the applets with regard to the two levels of variation?
3. Which epistemological obstacles occur?

3.2 Study design

After a pretest and a makeover the activities were employed within the framework of a qualitative study in grade 10 of two different secondary school classes ('Gymnasien') in Berlin, Germany. Both classes consisted of about 25 students and were taught by the researcher only for the time of the study within the conventional lessons. For each activity a block period was used and a single lesson was added in the end to write a short test and answering a questionnaire. In the first lesson of each block period the students worked nearly autonomously in pairs at one computer. For each activity a worksheet was provided so that one student was in charge of handle the applet while the other wrote the answers down (these roles were usually changed at some point). The second lesson was used to discuss the questions and answers with the whole class.

Per class two pairs of students where videotaped while working with the activities using the software 'ScreenFlow' which allows to record the students, their discussions, and the computer actions simultaneously. The discussion with the whole class was also videotaped. Additional material for analysis consists of the students' work sheets, the short test in the end and the questionnaires.

3.3 Methods of Analysis

It was expected that the diverse material for analysis enables the researcher to get a sophisticated impression of the students' train of thoughts. The main objective was to understand the students' negotiations in order to constitute mathematical sense. Therefore it was essential to analyze the recordings of the student pairs working on the activity sequentially using a microethnographical approach. For the analysis the researcher applies the principles of the tradition of the German interpretative classroom studies (Maier and Voigt 1991).

The main material for analysis consists of the recordings of the student pairs working on the activity. In a first step raw documents for analysis were created by going through the recordings sequentially. These documents consist of tables with the following columns: time, paraphrase, computer action, raw transcript, first interpretation. Also the information from the students' worksheets was integrated.

Based on these raw documents some episodes were chosen for final transcript and deeper analysis taking into account also the worksheets, questionnaire and test.

4 Results

In the following selected results relating to the interactive activities 'Dreiecksfläche' ('area of a triangle') and 'Einbeschriebene Rechtecke' ('inscribed rectangles') are presented. We focus on the question how the students construct mathematical sense using the applets and which mathematical terms they find and use during the learning process. Especially the impact of the level of metavariation on the learning process is of interest. Also some typical epistemological obstacles are pointed out.

4.1 Selected results and discussion – 'Dreiecksfläche' ('area of a triangle')

Distinction between 'stock and change'

The main difficulty of this activity was the distinction between 'stock of area' and 'change of area'. Stock and change had to be described in the situation as well as in the representation form graph. It appeared that the description within the situation was more difficult. In the situation one has to describe the aspect of change by analyzing the growth of area qualitatively whereas in the representation form graph this can be described by using terms like 'slope' or 'gradient'.

We present an excerpt of a transcript showing the discussion of the question 'Why is the graph shaped like this?' S1 stated that the graph increases at first and decreases after point C is passed. S2 does not agree. They discussion was much longer then the excerpt and the students got pretty loud. In the end another student (S3) intervened.

S2: But for me it does not decrease, it definitely increases. [..]

If it would decrease, it would go down again. [..]

S1: No, it declines. The ratio declines.

S2: No.

S1: Look, if it goes down like this.

S2: Ok, then let's say the ratio declines, but here *(points to the monitor)* it still increases. It also increases here and also there. [..]

S1: But proportional it decreases. [..]

S3: How about saying 'the slope decreases'?

S1: *(smiling)* That's right. Good.

For S2 the graph represents the stock of area whereas S1 tries to find information about change in the graph, but is not able to find appropriate terms for describing her observations in the situation. She speaks about some 'ratio' that decreases after point C is passed, but does not specify which ratio she means. S3 helps with the term 'slope' to describe the change on the graphical side.

Why is this difficult? In fact the applet shows the 'stock graph' (area function) next to the 'change graph' (triangle interpreted as piecewise linear function is the derivative). The stock graph is monotonic increasing, while the qualitative change of growth when passing point C is simultanously visible in the decrease of the derivative. In other words: 'Change in the stock graph' means 'stock in the change graph' and the learning activity connects both dynamically. This allows the students to communicate using visualizations.

Effects of metavariation

In fact the level of metavariation emphasizes local and global properties of the graph like monotony, inflection point, 'S'-form of the graph by being invariant. This presses the students to find explanations of what happens visually and leads their views to essential points.

But it was one specific obstacle of the applet that metavariation is a stronger visual impression than the first level of variation (moving point D). This led some students in a 'wrong' direction trying to explain the metafunction (=function between the triangle and the graph) rather than the area function. Therefore it was important to remind some students of the functional dependency itself by moving D and pointing out a pointwise view.

Epistemological obstacle 'slope in one point'

The next excerpt of a transcript shows one notable epistemological obstacle occurring in the learning process with the activity. The students try to describe the shape of the graph:

S1: I'd say the slope of the function changes.

S2: No, no, the slope, the function does not have a slope, because, if it would have one, it would be a line [..]

S2: *(moving C horizontally back and forth)* Look, [..] there is no slope, because the slope is different everywhere.

These students encountered the term 'slope' only in connection with 'slope of lines' so far. That is why S2 does not accept the term 'slope' in the case of the area graph. Moving point C (using the level of metavariation) seems to lead to another visual impression - the impression, that the slope 'is different everywhere'.

Student S2 formulates the need for an extension of the 'concept of slope' by herself. In class this could be used productively as a starting point for a discussion about the slope of a function like the given one, and about the possibilities to extend the concept of slope in a consistent way.

4.2 Selected results and discussion – 'Einbeschriebene Rechtecke' ('inscribed rectangles')

Three-stage approach

Part one of the activity (figure 4) is characterized by three stages:

First the students have to move the point P 'into the situation' to make an inscribed rectangle visual. This led to considerations about the *distinction between 'situation' and 'non-situation'*. E.g. when answering the question, if the area could also be 100, the students often moved point P to the coordinates (10 | 10) which is not on the line BC or mentioned that also the points (2 | 50) or (4 | 25) are not 'possible'.

Secondly – after fixing point P on the line BC – the students started to *gather values and to make conjectures* about properties of the area of the rectangle. Therefore they usually worked with approximate values for the coordinates of P and the area of the triangle.

Thirdly – after displaying the values for the coordinates of P and the area of the rectangle – the students start to *verify their conjectures* by working with more precise values.

The following excerpt of a transcript illustrates some aspects described above (note that the x/y-coordinates of P are denoted by P.x/P.y, and that the students have not displayed the values for the coordinates and the area yet):

S1: *(moving P to P.x=9, then to P.y=1 and finally choosing a position which is close to the coordinates (9 | 1). Note: P(9 | 1) is not on the line BC!)* Ok, here it is approximately 18. *(moves P to the approximate coordiates P(8.5 | 2.3))* The area doubles. *(moves P to P(8 | 3))* 3 times 8 is 24. [..]

S1: In the beginning it increases *(moves P to B and then in direction C)* The maximum is achieved when .. hmm *(moves P.x between 5.5 and 6.5)* [..]

S2: The perpendicular. [..] *(writes)* Area is maximal at the perpendicular through A.

S1: And grows non-proportionally.

S1 starts with some approximate values. His first conjecture seems to be, that the underlying function is a proportionality ('the area doubles' when moving P one unit further). His conjecture is rejected when realizing that P(3 | 8) does not lead to the triplication of the area. In the end he mentions the non-proportionality as a characteristic of the functional dependency.

Another (wrong) conjecture they make is that the maximum value is taken when P equals the intercept point of the perpendicular through A with the line BC. This results from a visual impression they get and maybe from the demand to use familiar geometric terms that seem to fit. Later they realize that this conjecture was wrong and revise their answer on the worksheet.

What is remarkable here is that the applet let the students *work mathematically in a certain sense*. To approach the situation the students gather dates and make conjectures in an uninhibited way, maybe because there are no negative consequences expected when exploring the interactive visualization autonomously. It was a strong impression that especially slow learners benefit from this fact. The conjectures are checked and/or revised basing on further experiences with the functional dependency.

Epistemological obstacle 'continuity'

Continuity is strongly related to properties of the real numbers (like completeness) and the idea of approximation when using decimal fractions. There were diverse situations in which the students faced observations that require mental concepts concerning continuity.

E.g. when making conjectures about the change of the area in applet one (figure 4) statements like the following occurred:

"Here it [the area] is zero und then it grows more and more", "How do you know that it grows?", "Anyway it becomes larger than zero. Thus it grows at first.", "No, I think the area is always the same."

The fact that the area changes continuously does not seem to be intuitively obvious for all students. It is not probable that the student who states, that the area does not change, is aware of the fact that this means a jump discontinuity of the area function.

Closely related to the concept of continuity is the concept of approximation. It was observed that the students usually started by gathering some approximate (imprecise) values when searching for the maximum. They were not of aware of the necessity to choose more and more precise values while coming closer to the maximum. E.g. most students conjectured at first that the maximum is taken in P(6|6) and the maximum area is 36. One pair of students confirmed this by moving P approximately on the coordinates P(5.2|7.1) and stating that 5 times 7 is 35 and therefore lower than 6 times 6. Note that the maximum of 37.5 is taken in P(5|7.5) and that 5.2 times 7.1 results in an area of 36.92 which is in fact larger than 6 times 6!

It was interesting to observe that the applets allow the students to experience continuity in an enactive way, and provoked them to find terms to describe this phenomenon. E.g. one pair of students discussed the question if the value 25 is taken or not when exploring applet two (figure 5, left). One student thought it will not be taken, because it did not appear as a decimal fraction beneath the applet. The other student does not agree. In the end they formulated continuity in terms of: "Each value appears two times. *(moves P back and forth)* It is always possible to move the point a tiny bit." This formulation was made possible using the graphical representation dynamically. The observation of the displayed data was not appropriate for the establishment of mental images about continuity, because the decimal fractions are finite and do not reflect the property of completeness of the real numbers.

Metavariation and predicative thinking

Especially in this activity the level of metavariation is difficult to understand in a functional-dynamic way, because the metafunction (assigning the triangle ABC to the area graph) is two-dimensional depending on the points B and C. Moreover it is difficult to simulate the change of the area mentally within a given situation because there are two underlying functional dependencies: the dependency between the coordinates of P (which is a linear function) and the dependency between the x-coordinate of P and the area F(x) (which is quadratic).

Metavariation is easier to understand when finding a predicative description of the situation. A predicative description considers the structure and invariants. It aims at the properties and their interrelations (Schwank 1996).

It seemed to be a specific obstacle of the applet that it strongly suggested a functional-dynamic view instead of a structural view.

In fact the students formulated both views. E.g. one student solution to the question 'Give two examples for triangles so that the rectangle with maximum area is 15. Describe how you found these triangles.' was:

"On the x-axis you need a number that results in 60 when multiplied with n or you take an arbitrary point on the x-axis and move the point on the y-axis as long as the maximum is 15."

Their structural approach is only understandable when watching the recording. It is meant, that the x-coordinate of B multiplied with the y-coordinate of C must result in 60, because the maximum area is one forth of the rectangle formed by completing the triangle ABC to a rectangle. Their second description is a functional-dynamic one basing on the use of the applet. It does not integrate an object view of the situation.

Once the students have found an appropriate predicative description it enabled them to think reversible in sense of Piaget (1972). One pair of students used the following predicative description when characterizing the location of the maximum: 'The maximum is found by reducing the coordinates of P by half and multiplying these values.' To answer the question of how they found the triangles, so that the maximum of the inscribed rectangles is 15, they reversed this operation mentally by saying "For example one can [compute] 15 divided by 3 [..] then you get 5 and you take 5 times 2 and 3 times 2 and get <u>these</u> values [x-coordinate of B and y-coordinate of C]".

5 Summary

The results show, that the interactive activities are practicable and effective means to be used within the framework of propaedeutics of calculus. Students discovered global and local properties of functions, built terms related to concepts of calculus on their own and integrated familiar terms. For example students explore the distinction between stock and change by themselves using the dynamic visualization. Metavariation relates to the object view of a function and emphasizes global and local properties of the dependency. But metavariation also evoke some obstacles specific for the applets, because of the strong visual impression and one need to be aware of this fact when using the activities in class.

When exploring the applets students worked mathematically in a certain sense: gather data, making conjectures, test conjectures. The applets enabled them to work in this way without negative consequences which had a positive effect on slow learners.

Also some notable epistemological obstacles like 'slope in one point' or 'continuity' appeared which can be used productively in the further learning process.

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