3D Dynamic Geometry Tools – Helper or Obstacle?

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Abstract

The 3D dynamic models can be a great help in teacher's effort to support students' spatial abilities, their understanding concepts of solid geometry and core of 3D problems. However, handling such a tool may be a difficulty by itself and an undesirable obstacle for students. Tools need to be easy to use and models for students need to be well-prepared. We see the use of these tools in a few aspects: Illustration and demonstration of spatial problems that are difficult or time-consuming to describe via planar views or solid models, explanation of students' mistakes and clarification of the core of the problem itself instead of planar static manifestation of the correct answer via picture or verbal description, power to attract and maintain students' attention and to provide the correct answer to the problem or task automatically without teacher's effort, visualization of the answer to the problem and power to check the correctness of the individual steps in constructing 3D objects, support of spatial imagination related to motion and power to facilitate and/or automate teacher's creation of didactic printed materials and tests including correct answers to them without necessity to solve each of the variants of the test individually. The aim of this paper is to present examples of such dynamic geometry models that are easy to use and explain for teachers as well as for students.

Keywords

3D systems Spatial abilities Constructions Tests Polyhedra

1 Introduction

Are you trying to improve your students' spatial abilities? Do you create additional materials and tests for students? Are you usually short of time to train general skills and to reveal and explain students' mistakes during lessons?

Though manual manipulation is the best method for acquiring spatial abilities, the solid models often cannot be flexible enough. Though construction of 3D objects using pencil, paper and knowledge of projection methods trains ability of abstraction well, it takes a good deal of time and it absorbs students' effort in mastering the projection itself instead of solving the 3D problem given.

Cultivation of different students' spatial abilities requires a lot of teacher's effort and 3D dynamic geometry tools bring a great help. However, for some students handling such a tool may be a difficulty by itself. A 3D model cannot be touched. It is not a solid model, it is an abstraction. Keeping all the above mentioned in mind, we present some didactic 3D dynamic models in this paper.

2 Using 3D models

2.1 Demonstration and visualization of 3D concepts

There are a lot of common students' misunderstandings that can be easily explained using 3D models. (For example, the cylinders: oblique, right and circular cylinder – students often consider only a right circular cylinder. Being asked to demonstrate an oblique circular cylinder they just rotate the previously mentioned.)

Polyhedra properties, especially tetrahedron properties, usually are not very clear from the solid model as they need to be demonstrated on a lot of different tetrahedra.

The concept of inversion in space can be explained using planar inversion (which is also easy to explain using plane dynamic geometry tools). But students have never had an opportunity to see an example of inversion in real world, so it is difficult for them to imagine. Though constructions that use inversion are usually too complex and thus not very clear and easy to follow, we will demonstrate one quite easy example of using inversion.

Inscribed solids. Some students don't understand well the concept of inscribing one solid into another. If they are shown the process of its construction, it may clear up the idea.

2.2 Explanation of problems, getting ideas

As mentioned above, training of general spatial abilities is very important for solving solid geometry problems. Being not focused on particular curriculum topic and their intention being to attract and maintain students' intensive concentration on imagining in 3D, some quizzes, tests and games are created.

Cubes

A cube of volume of 4^3 units is divided into unit cubes. How many of them are cut by plane perpendicular to cube's space diagonal and passing through a given point? Try to find the position of plane for which the number is maximal.

There are examples of such planes and answers to the particular configurations shown in Fig. 1. The model is flexible and enables us to modify the position of the plane (the line that the plane is perpendicular to can be rotated) and thus to make the task less or more difficult. It shows the correct answer dynamically. Not only the model gives an answer to the particular task, but it can give a hint for solving more general tasks.

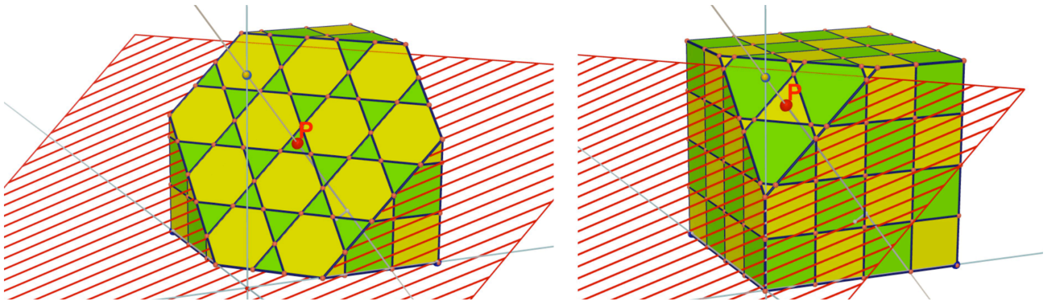

Figure 1. Planes cutting given cube.

Volumes 1

Dynamic geometry tools are very useful for discovering facts or inventing unknown ideas. Then the proof must be given, of course, but the finding of new ideas is challenging and it gives students a motivation.

One of common tasks in solid geometry is finding volumes of solids. Students are taught a lot of formulas and use them. While the slowest students struggle hard to solve simple tasks, the best tend to be bored. They should be given some interesting problems.

Let's derive the formula for the volume of a cut triangular prism (the cutting plane intersects all the three parallel edges of the prism with base triangle *ABC*). We will take a dynamic geometry model for help. Simple blind guessing and examining the volumes is useless, of course. But some ideas about the centroid may give a hint. The task to find the planes that halve the prism (divide it in two parts of equal volume) can be presented at first. A similar problem with a rectangular parallelepiped that is easier to solve can help, too.

The line passing through the centroid of the base triangle parallel to the edges of the prism will be important. Let *M* be the point of this line and *K*, *L* points on the two edges of the prism. The model shows that *M* being fixed, the volume of the cut prism (cutting plane being *KLM*) is constant. But, now, our new idea must be proved!

2.3 Explanation of correct answers

Volumes 2

The plane *BDE* cuts the tetrahedron *ABDE* from the box (rectangular parallelepiped) *ABCDEFGH*. The volume of the tetrahedron is *V*. Find the volume of the tetrahedron *A B*D*E** that is cut from the quadrant *ABDE* by the plane passing through the point *G* and parallel to the plane *BDE*.

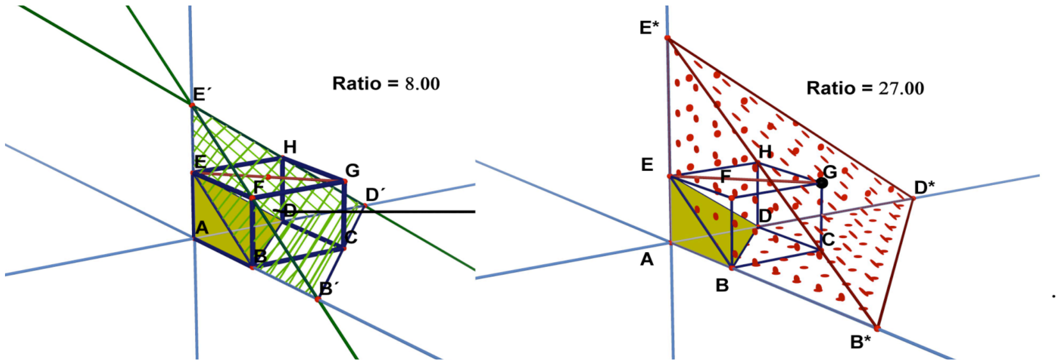

Figure 2. Volumes of tetrahedra.

Usual (wrong) students' answers are $2V$, $8V$. The model reveals not only the answer, but it helps to find the reasoning, too.

Plane sections of solids

(Fig. 3) Plane sections of polyhedra are popular solid geometry tasks. Dynamic geometry tools can automate the process of creating sets of tasks of required difficulty. Demonstrated models can be used in two ways – as dynamic interactive models that allow students to solve given tasks and then verify their answers, and as the tool that teachers can use for creating printed tests as well.

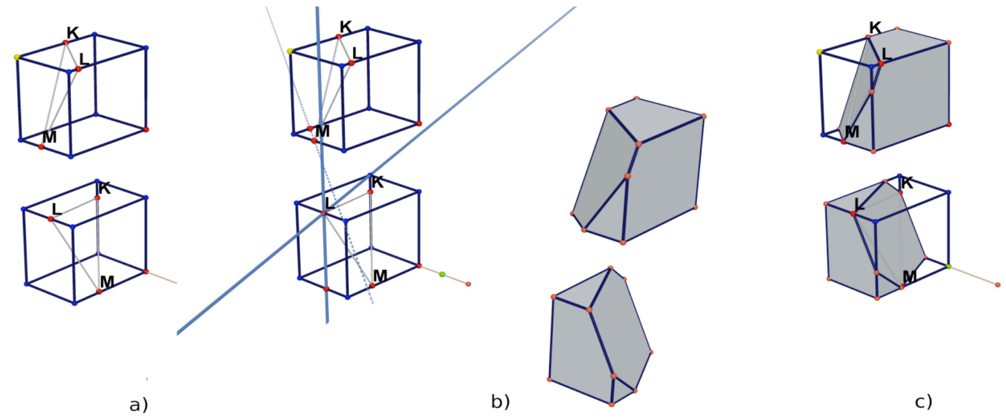

Figure 3. Given task (a), first steps of solving the problem (b) and checking the answer (c).

2.4 Visualizing 3D constructions and checking the answers

Ruler and compasses constructions in plane are usually regarded as difficult. Such tasks in space are even more difficult. Not only they require a good deal of spatial imagination, it is also the process of projection that makes it worse. Carrying out analysis of the problem from the diagram is not easy even if the sketch is clear enough. The tools of dynamic geometry can help students to concentrate on the problem instead of the drawing. Moreover, they allow students to really construct the model and examine it finally.

This way of using dynamic geometry tools by students is probably the most difficult one as they have to find proper tools and use them aptly. It is usually useful to provide them with the scene with the given elements fixed. Yet, only constructions requiring few steps are appropriate as the lesson tasks, the more complex ones – if ever – should be let for homework.

Tetrahedra

Construction of an orthocentric tetrahedron makes students to consider spatial relations. Following tasks are easy to solve using dynamic systems.

Task 1. The face ABC of orthocentric tetrahedron is given. Find the locus of the vertices D.

Task 2. Let the vertices A, B and orthocenter V be given. Construct the orthocentric tetrahedron.

Task 3. Construct an orthocentric tetrahedron with three vertices on two perpendicular skew lines – edges – given. (It is easy to solve.)

Task 4. Find the tetrahedra of maximum and minimum volume if lengths of the edges AB, CD on the two given skew lines are $|AB| = p$, $|CD| = q$. Explain your answer. (Dynamic geometry shows immediately the fact, that this volume is constant. Nevertheless, the reasoning of this fact requires some idea.)

Common tangent plane to three spheres.

Construct any (all) plane(s) tangent to the three spheres given.

Abstract model allows us to find all such planes with their points of tangency. In the plane given by the centers of spheres (if centers are not collinear) we can find four lines passing each through one triple of homothetic centers of the spheres given. We can find up to eight planes (if pairs of the spheres have no intersection). They are given by the choice of one of these four lines and a tangent line to any of the three spheres perpendicular to the line chosen.

Common tangent sphere.

We have chosen one of the tasks regarding tangent spheres and planes. Particular position of given elements in this case makes the task clearer and still very informative. Above that, it shows the prospect of dynamic tools in solving problems.

Task. Two externally tangent spheres of different radii and their common tangent plane are given. Find the locus Λ of centers of all spheres tangent to two given spheres and to a given plane. Find the loci Λ_1, Λ_2 , Λ_3 of points of tangency on given surfaces, too.

Being not experienced in solving such tasks, one may not have a clue how the desired locus looks like. But the construction of such a center is easy (we can construct a sphere of particular radius). Then using the "Locus" tool we get the hint: The locus seems to be an ellipse. By analogy with the planar problem, we can reveal the fact that locus of the centers of spheres tangent to a given plane and a given sphere is a circular paraboloid (two paraboloids, in general). Its axis is perpendicular to the plane given and passes through the given sphere center. Thus the desired locus is a curve of intersection of two paraboloids with parallel axes. It is interesting to observe that this intersection is planar. We can show it using analytic geometry or from the following theorem of projection: "The elliptic section of a paraboloid is orthogonally projected into the plane orthogonal to the axis of the paraboloid into a circle." We see that the plane ρ of axes of the paraboloids is the plane of symmetry of this section. Plane sections of paraboloids by a plane perpendicular to ρ passing through "the lowest" and "the highest" crossing points A, B (if they exist) are ellipses with major axis AB. But the projection of segment AB into a given plane is the diameter of both the circles to which these ellipses are projected. Thus these circles coincide and the ellipses of the two planar sections coincide, too, being thus the curve of intersection of the paraboloids.

Moreover, the projection of this ellipse into the given plane – circle k – is the locus of points of tangency of spheres with the plane given.

Finally, each point of tangency of a given sphere α and the found sphere κ is their homothety center. Homothety transforms a given plane (tangent to κ) to a parallel plane tangent to α that touches α in point P. It is known that the orthographic projection of a circle (not passing through P) from the point P is a circle. Thus the loci of points of tangency on the spheres given are circles.

This pretty complicated deduction can be shortened effectively using inversion. Transforming the given spheres into parallel planes that are tangent to the sphere that is the image of a given plane transforms our problem and simplifies it. We can easily construct circles that are the loci of tangency points of tangent spheres (images in inversion). Using this inversion again, we get the circles of points of tangency on the given spheres and plane. Nevertheless, a locus of centers of spheres – an ellipse – has to be constructed using tangent points next.

3 Creating printed tests and materials

As mentioned above, models can help teachers to assemble, modify and print out tests or worksheets for students. Models may contain nets of polyhedra, the above mentioned plane sections of prisms and pyramids etc. We show two models for testing and training general spatial abilities.

The first test tasks are to complete the image of a given cube with glued small cube inside in different view. (Fig. 4)

In the last one the students are asked to tilt a given cube (to imagine tilting it) along its edges repeatedly and then draw it (and patterns on its sides) in its new position. (Fig. 5)

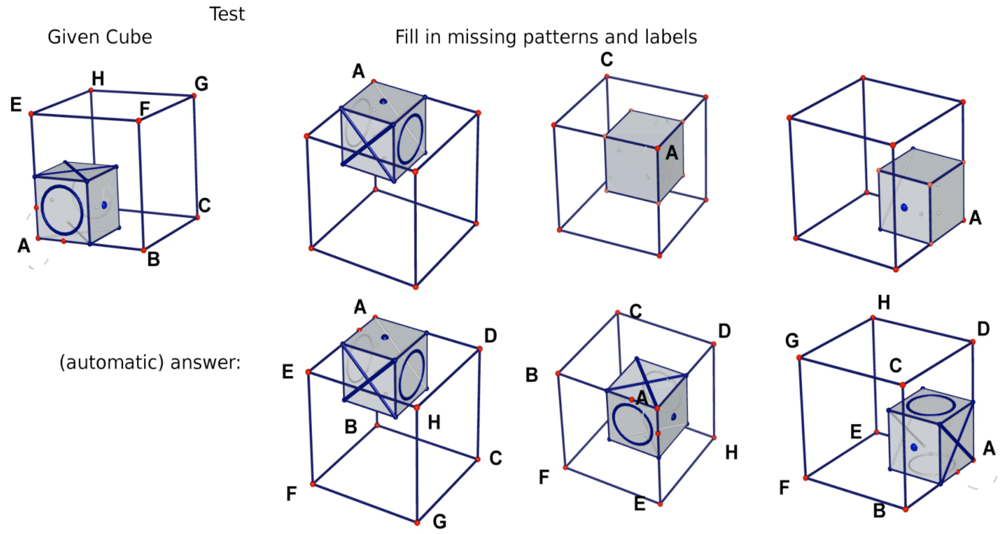

Figure 4. Different projections of given shape.

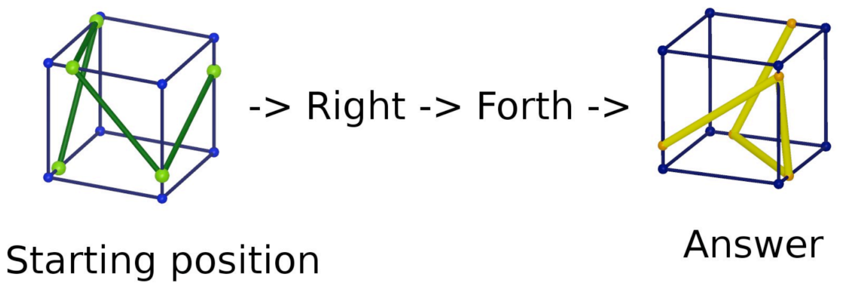

Figure 5. Rotating the cube.

4 Conclusions

Dynamic 3D models widen teachers' possibility to follow their students' abilities and needs more individually. But using models must be only one of many methods and it is vital to keep in mind the difference in students' attitude to computers and to abstract 3D models. Some students – usually the high achievers – may be really taken, whilst for their slower peers the necessity to manipulate with a virtual model may be only another obstacle. Such students must be given printed materials and – especially – the real solid models instead as often as possible.